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6. AUTHOR(S)
J. Daniel Cobb

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Department of Electrical and Computer Engineering
University of Wisconsin
1415 Johnson Drive
Madison, WI 53706-1691

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This report describes the results obtained during the two years of funding extending from 1/88 to 1/90 in support of our project entitled "The problem of Robust Compensation for Systems with Unmodeled Dynamics." The project has consisted of several lines of research which are quite distinct, but which show great promise toward combining them into a single comprehensive theory. Our goal has been to explore the problem of designing feedback control systems that are insensitive to the presence of high-frequency dynamics not accounted for explicitly in the mathematical model of the plant. Some of our previous work suggests that it is possible to design controllers which simultaneously stabilize a given nominal system as well as a large class of small singular perturbations of the system. Attached are six papers summarizing work which has been supported all or in part by the present grant and which either have appeared, have been accepted, or have been submitted for publication.

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The Problem of Robust Compensation for Systems with Unmodeled Dynamics

Final Report

Principal Investigator:

J. Daniel Cobb
Department of Electrical
and Computer Engineering
University of Wisconsin
1415 Johnson Drive
Madison, WI 53706-1691

Abstract

This report describes the results obtained during the two years of funding extending from 1/88 to 1/90 in support of our project entitled "The problem of Robust Compensation for Systems with Unmodeled Dynamics." The project has consisted of several lines of research which are quite distinct, but which show great promise toward combining them into a single comprehensive theory. Our goal has been to explore the problem of designing feedback control systems that are insensitive to the presence of high-frequency dynamics not accounted for explicitly in the mathematical model of the plant. Some of our previous work suggests that it is possible to design controllers which simultaneously stabilize a given nominal system as well as a large class of small singular perturbations of the system. Attached are six papers summarizing work which has been supported all or in part by the present grant and which either have appeared, have been accepted, or have been submitted for publication.

FINAL REPORT FOR AFOSR GRANT #88-0087

This report describes the results obtained during the two years of funding extending from 1/88 to 1/90 with regard to our project entitled "The Problem of Robust Compensation for Systems with Unmodeled Dynamics." The project has consisted of several lines of research which are quite distinct, but which show great promise toward combining them into a single comprehensive theory. Our goal has been to explore the problem of designing feedback control systems that are insensitive to the presence of high-frequency dynamics not accounted for explicitly in the mathematical model of the plant. Some of our previous work suggests that it is possible to design controllers which simultaneously stabilize a given nominal system as well as a large class of small singular perturbations of the system. We say such a controller is "robust" with respect to the certain class of unmodeled dynamics.

Attached are six papers summarizing work which has been supported all or in part by the present grant and which either have appeared, have been accepted, or have been submitted for publication. Two have already appeared in the *IEEE Transactions on Automatic Control*, one is scheduled to appear in the same journal in May, one has appeared in the *Proceedings of the 27th IEEE Conference on Decision and Control*, and one is presently under review. In addition, a summary page is included describing a paper presented at the *SIAM Conference on Control in the 90's* (5/89, San Francisco). We also are in the process of writing a journal article describing the work carried out in the the final stage of the project. The manuscript should be available within the next two months. Part of the research effort over the past two years has been carried out by my graduate student Mingde Tan as part of his Ph.D. research. His dissertation should be available sometime this spring.

Reference [1] contains work which was partially carried out during the funding period and which was instrumental in establishing the direction outlined in the original grant proposal. Our basic idea was to explore the role of parasitics in the performance of automatic control systems, without having to resort to explicit representations of specific parasitic effects. It was our desire to develop a comprehensive theory of compensator design which would guarantee performance in the presence of a large class of possible parasitic effects. Reference [1] contains preliminary results directed at this goal.

Along somewhat different theoretical lines, but with a similar class of engineering problems in mind, references [2]-[4] constitute initial attempts, carried out with the aid of a colleague here at Wisconsin, Professor Chris DeMarco, to characterize the geometric structure of the class of system perturbations under which a control system retains stability and perhaps other performance characteristics. Contained in [2]-[4] is a thorough treatment of the case where system order is constant (the nonsingular case). Our more recent work may be viewed as extending these results to the case where parameter variations can cause changes in system order (the singular case).

Reference [5] summarizes the work carried out by my graduate student, Mingde Tan, and me over the initial phase of his dissertation research (the first year of the funding period). Our principal idea was to study the effects of small system perturbations on internal closed-loop stability. Ultimately, we wish to examine many of the recent robust control theories of Vidyasagar, Zames and Francis, Stein and Doyle, and others in terms of internal closed-loop system behavior. For example, suppose a plant model is given and a corresponding compensator is designed using some methodology such that the closed-loop configuration is input-output stable. The design methods

of the researchers just mentioned typically guarantee that, for a certain class of perturbations of the plant (and sometimes the compensator), the corresponding perturbed closed-loop system is also stable in an input-output sense. It would be highly desirable to know whether the perturbed closed-loop system is internally stable as well. From our perspective, it is especially crucial that the compensator be insensitive to perturbations corresponding to unmodelled high-frequency dynamics.

In order to formulate this problem precisely, one must first develop an understanding of how internal system structure is affected by perturbations of the transfer function. To this end we have developed a "perturbational" analogue of the standard state-space realization theory for rational matrices In our theory, families of rational matrices are considered (either convergent sequences or continuous parametrizations); it is desired to find corresponding (convergent) families of state equations which realize the given transfer matrices. We have obtained results that show, for example, that every such family of transfer matrices has a realization and that "minimality" of a realization can be related to both the degrees of the given :ational functions as well as to controllability and observability of the realizations themselves. In short, we have succeeded in developing the perturbational analogues of the standard results concerning realization of a fixed rational matrix. This body of results is among our main accomplishments in the past two years.

Based on our understanding of the fundamental issues surrounding internal realizations of perturbed transfer functions, we have spent a large portion of the past year exploring the relationships between robust input-output stability and robust internal stability. We have succeeded in establishing simple conditions under which internal as well as input-output stability is

robust to parasitic model uncertainty. These results form the second half of Mingde Tan's dissertation and are presently being organized for journal publication.

In addition to the work with Mingde Tan, I personally have continued the effort I began in 1986 which addresses questions similar to those described above related to internal behavior of robust closed-loop systems. The culmination of this effort so far is the paper [6] which shows that robust design methodologies necessarily must incorporate some internal system information; the exact form of the "minimal" internal information is as yet unknown. A recent breakthrough has been summarized in [7] which was presented in May 1989 at the SIAM Conference on Control in the 90's in San Francisco. (There were no published Proceedings for this conference.) For a certain large class of linear systems, it is now possible to precisely characterize the family of singular perturbations under which closed-loop stability is retained.

Overall, our two-year research effort has been fruitful, answering many important questions and leading to new ones. In the final analysis, the entire body of results we have generated is not as coherent as we original hoped for; however, we feel that, given the time-limit set for the project, the progress made in each research direction more than compensate for this fact.

[1] J. D. Cobb, "Toward a Theory of Robust Compensation for Systems with Unknown Parasitics," *IEEE Transactions on Automatic Control*, Vol. 33, No. 12, December 1988.

Toward a Theory of Robust Compensation for Systems with Unknown Parasitics

J. DANIEL COBB, MEMBER, IEEE

Abstract—We consider the problem of designing a robust compensator based on a plant model with order uncertainty. The uncertainty is characterized mathematically as a class of generalized singular perturbations of the plant. This paper considers the case of static compensation. A necessary and sufficient condition is established under which actual closed-loop behavior is close to that predicted by the plant model under sufficiently small singular perturbations. The condition is shown to be generic.

I. INTRODUCTION

THE problem of robust compensation may be roughly stated as that of designing a good controller for a given physical system on the basis of a model which contains less than complete information about that system. The resulting closed-loop configuration should exhibit reasonable performance in spite of the uncertain aspects of the system. In the strictest sense, every model contains uncertainty; hence, any good controller design should address the issue of robustness.

Among the many types of robust control theories appearing in the literature is the asymptotic approach. Typical results in this area guarantee reasonable closed-loop performance under sufficiently small perturbations of a nomial model (e.g., variations in the coefficients of a single differential equation). Although only local in nature, such results are often a first step in developing a global theory where an explicit characterization is attained for classes of systems which can be simultaneously compensated. The results of this paper fall into the asymptotic category.

It is possible to view most asymptotic robustness theories within a common mathematical framework. Let $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{I}$ be topological spaces, and let $\mathcal{R} \subset \mathcal{P} \times \mathcal{Q}$ inherit subset topology. $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{I}$ correspond to the sets of all possible models of plants, compensators, and closed-loop systems, respectively. The topologies on $\mathcal{P}$ and $\mathcal{Q}$ are chosen so that small perturbations characterize measurement error inherent in developing each model; small perturbations in the topology of $\mathfrak{I}$ reflect tolerable closed-loop performance error. If $\mathcal{R}$ is interpreted as the class of all plant–compensator pairs which lead to closed-loop systems that are well-defined and which satisfy any additional constraints present in the design problem, we may naturally define the *loop-closing map* $\mathcal{C}$: $\mathcal{R} \rightarrow \mathfrak{I}$ which takes each plant and compensator into their corresponding closed-loop configuration. Many robustness questions then reduce to that of finding the points of continuity of $\mathcal{C}$. In other words, we wish to characterize the class of all plant–compensator pairs such that small perturbations of each pair result in small perturbations in the closed-loop system.

We now examine various existing theories which lie within the asymptotic framework. The most obvious body of such results centers around the well-known fact that, for state-space models, the parameters of the closed-loop system are continuous functions of the open-loop plant and compensator parameters. For example, if we let $\mathcal{P}$ be the set of all matrix triples $\xi = (A, B, C)$ and $\mathcal{Q}$ consist of all feedback matrices K, and if we combine ξ and K in a standard way, then $\mathcal{R} = \mathcal{P} \times \mathcal{Q}$ and $\mathfrak{I}$ consists of triples $\mathcal{C}(\xi, K) = (A + BKC, B, C)$. Adopting Euclidean topology on $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{I}$, it follows that $\mathcal{C}$ is continuous everywhere, i.e., every compensator is robust relative to every plant. One immediate consequence of this observation is that closed-loop eigenvalues are continuous functions of plant and compensator parameters; hence, every stable closed-loop configuration remains stable under sufficiently small parameter variations. These facts are used routinely in many control system analyses without explicit mention. It should be noted, however, that the perturbations considered here do not alter either plant or compensator order. Therefore, this approach alone is inadequate when dealing with order uncertainty.

The main body of existing results that does deal with order uncertainty in an asymptotic setting can be broadly termed singular perturbation theory (see [1]–[3]). Here a typical analysis treats a parametrized system of the form

$$\begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix} \dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$y = [C_1 \;\; C_2] x \tag{1}$$

with A_{22} stable and seeks to achieve some closed-loop performance criteria for all sufficiently small $\epsilon \geq 0$. (In this case, we might take $\mathcal{P} = [0, \infty)$.) A major drawback with this approach is that explicit knowledge of the parasitic structure giving rise to order uncertainty is assumed. If more than one perturbation (1) need to be considered, serious problems may develop. For example, the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = [-1 \;\; 1 \;\; 0] x \tag{2}$$

is nominally ($\epsilon = 0$) unstable, but can be stabilized with the static compensator $u = 2y$. The perturbed system ($\epsilon > 0$) is also stabilized by the same compensator for sufficiently small ϵ. Setting $\epsilon = 0$, premultiplication of (2) by the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

yields an equivalent system equation which may in turn be

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The author is with the Department of Electrical and Computer Engineering, University of Wisconsin-Madison, Madison, WI 53706-1691.

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perturbed according to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & -2 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u$$

$$y = [-1 \ 1 \ 0] x. \tag{3}$$

In this case the compensator $u = 2y$ yields a perturbed closed-loop system having a pair of eigenvalues $\lambda_{1\epsilon}$ and $\lambda_{2\epsilon}$ with $\text{Re}\,\lambda_{i\epsilon} \to +\infty$ as $\epsilon \to 0^+$. Such divergent behavior does not coincide with any reasonable definition of small perturbations in $\mathfrak{I}$. We may therefore conclude that examination of a single parasitic effect is in general not sufficient to guarantee robustness of a compensator with respect to other order uncertainties.

Additional singular perturbation results include the multiple time-scale extensions [7] and [8] and the robust compensation theorems of [5]. Multiple time-scale techniques suffer from the same drawback as single time-scale analyses based on (1) in that they assume an explicit knowledge of parasitic structure. Also, much less is known about the ϵ-dependence of the time response of multiple time-scale systems than in the single time-scale case.

In [5] it is shown that any compensator having a strictly proper transfer function matrix, which stabilized (1) with $\epsilon = 0$, also stabilizes (1) when $\epsilon > 0$ is sufficiently small. Furthermore, it is shown that the corresponding family of closed-loop transfer matrices converges uniformly on compact subsets of the right-half complex plane as $\epsilon \to 0^+$. These results thus provide a means for robustly compensating a system in the presence of a large class of possible perturbations. One drawback to this theory is that only single time-scale systems (1) are treated. In practice, a much larger class of perturbations may be required to model all relevant effects. Additional problems are that the results of [5] do not take into account uncertainties in the compensator model and that uniform convergence on compact sets in $\mathbb{C}$ is difficult to relate to time-domain performance of the system.

Another notable asymptotic robustness theory is that of [6] where the graph topology is introduced. Let $\mathcal{P}$ and $\mathcal{Q}$ each be the space of all rational matrices, $\mathfrak{I}$ the space of strictly proper and stable rational matrices, equipped with the H_∞ norm, and $\mathcal{R} = \mathcal{C}^{-1}(\mathfrak{I})$. The graph topology is the weakest topology on $\mathcal{P}$ and $\mathcal{Q}$ under which $\mathcal{C}$ is continuous. We have shown in [9], however, that singularly perturbed systems generically do not converge in the graph topology; hence, in this sense, robust compensation in the presence of order uncertainty is unattainable.

In view of the shortcomings of the existing asymptotic techniques, we wish to propose a framework as well as some preliminary results for an alternative robustness theory which will be taken into account: 1) multirate and other relatively unexplored classes of singular perturbations; 2) the necessity of dealing simultaneously with a large class of system perturbations, each corresponding to a possible higher order model; and 3) time-domain behavior of the closed-loop system. Although treatment of 1) and 2) seems on the surface to be a formidable task, we will see that it is possible to approach the problem in a roundabout way, thus avoiding having to explicity characterize all possible parasitic phenomena. We feel that the inclusion of 3) is a desirable feature for any good robustness theory, since the goal of system design must ultimately be satisfactory closed-loop time response. In view of this fact, a time-domain approach has certain advantages over frequency domain techniques, since the relationship between time response and frequency-domain behavior can be rather complex.

Before becoming too engrossed in technicalities, we will briefly describe (in rough terms) the problem we wish to address. Consider the system

$$\xi: \quad \begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{4}$$

where E, A, B, are real matrices with E and A square. We assume that (4) exhibits existence and uniqueness of solutions for each initial condition x_0 and each input function u; from [17] we know that this is equivalent to

$$|sE - A| \not\equiv 0.$$

Such systems have been studied extensively (e.g., see [14]–[16]), and are referred to as singular when E is singular and regular otherwise. The polynomial

$$\Delta(s) = |sE - A| \tag{5}$$

may be considered the characteristic polynomial of (4) and its roots the eigenvalues of ξ. An important property of singular systems is that small perturbations in the entries of E and A can change the system order; one example of this phenomenon is (1).

Suppose we wish to find a compensator of the form $u = Ky - v$ which is robust with respect to perturbations in E, A, B and C. Since we are inevitably interested in time response, we might ask which compensators result in a closed-loop system whose time response varies continuously with E, A, B, and C, regardless of the perturbation. Unfortunately, it is easy to show that for any K there exist perturbations in the system matrices that yield divergent behavior in the closed-loop system trajectories for some initial conditions. A more meaningful problem can be formulated by first observing that not necessarily all perturbations in the matrix entries of (4) are physically realistic. For example, a simple RC circuit consisting of a single resistor, capacitor, and voltage source may be modeled as

$$\epsilon\dot{x} = -x + u$$

$$y = x \tag{6}$$

where x is the capacitor voltage, $R = 1$, and $C = \epsilon$. Positive ϵ makes perfect physical sense, and it seems reasonable to try to design a compensator based on the low-order model corresponding to $\epsilon = 0$. On the other hand, if ϵ is negative, the system engineer could not expect to produce a robust compensator without first being aware of the negative capacitance and then using an appropriate higher order (in this case, first-order) model.

A simple way to characterize physically meaningful perturbations in the plant is to look at their effect on plant trajectories for various inputs and initial conditions. For example, in (6) an initial condition $x_0 = 1$ yields $x(t) = e^{-t/\epsilon}$ which converges on compact subintervals of $(0, \infty)$ as $\epsilon \to 0^+$, but diverges as $\epsilon \to 0^-$. Strictly speaking, we are really not saying as much about perturbations of (4) which can occur in the physical world as we are about those perturbations which are consistent with the measurements taken while formulating our plant model; a system model is good only if it is capable of predicting the behavior of the actual physical system.

We may now state our definition of asymptotic robustness more precisely. For a given plant of the form (4), a compensator is robust if all perturbations in both the plant and compensator, which bring about only small variations in the trajectories of each system individually under all inputs and initial conditions, result in only small variations in the closed-loop system trajectories. The meaning of the phrase small variations will be precisely defined in Section III. In the same section we will see that our approach implicitly incorporates the idea that small system variations should correspond to only small changes in system parameters.

II. Preliminaries

In this section we summarize the constructions of [10], [11], [13], and [14] which are pertinent to subsequent developments. Let

$$\Sigma(n, m, p) = \{(E, A, B, C) \in \mathbb{R}^{n(2n+m+p)} \mid |sE - A| \not\equiv 0\}$$

and let $\mathcal{L}(n, m, p)$ be the corresponding quotient manifold (see [18]) determined by the equivalence

$$(E_1, A_1, B_1, C_1) \approx (E_2, A_2, B_2, C_2) \text{ iff } C_1 = C_2 \text{ and } \exists \text{ nonsingular } M \text{ s.t. } ME_1 = E_2,\ MA_1 = A_2, \text{ and } MB_1 = B_2. \quad (7)$$

(The arguments n, m, and p will be dropped when clear from context.) We choose the equivalence relation (7) because premultiplication by M has no significant effect on the system representation. Indeed, premultiplication by M merely performs elementary row operations on the system of scalar equations (7). Hence, we are merely identifying systems formed from each other by reshuffling the equations. We do not wish to identify systems which are related by a coordinate change on the state variable x, since this would reduce the system space to one consisting of input-output descriptions. Our intention is to produce results which exploit internal information.

The equivalence class containing $\sigma = (E, A, B, C)$ is denoted $\xi = [E, A, B, C]$. In this case, we say σ represents ξ. Let

$$r = \text{ord } \sigma = \text{ord } \xi = \deg \Delta$$

where Δ is the characteristic polynomial (5) of ξ, and note that a unique matrix C is determined by each $\xi \in \mathcal{L}$. A sequence $\xi_k \in \mathcal{L}$ converges weakly to $\xi \in \mathcal{L}$ ($\xi_k \xrightarrow{w} \xi$) if $\xi_k \to \xi$ in manifold topology. Since $\mathcal{L}$ is a quotient manifold, the natural projection $(E, A, B, C) \mapsto [E, A, B, C]$ is continuous with respect to weak convergence. Conversely, we have shown in [10] that, for each convergent sequence $\xi_k \xrightarrow{w} \xi$ in $\mathcal{L}$, there exists a sequence $(E_k, A_k, B_k, C_k) \to (E, A, B, C) \in \Sigma$ such that $[E, A, B, C] = \xi$ and $[E_k, A_k, B_k, C_k] = \xi_k$ for every k.

Let $\xi_k \xrightarrow{w} \xi = [E, A, B, C]$ with E singular. In [11] is shown that there exist nonsingular matrix sequences $M_k \to M$ and $N_k \to N$ such that

$$M_kE_kN_k = \begin{bmatrix} I_r & 0 \\ 0 & A_{fk} \end{bmatrix}, \quad M_kA_kN_k = \begin{bmatrix} A_{sk} & 0 \\ 0 & I_{n-r} \end{bmatrix} \quad (8)$$

where $r = \text{ord } \xi$, $A_{sk} \to A_s$, and $A_{fk} \to A_f$ with A_f nilpotent. For sufficiently large k, the matrices A_{fk} and A_{sk} are unique up to a similarity transformation. For a constant sequence, the decomposition (8) reduces to the Weierstrass decomposition for matrix pencils (see [17])

$$\text{MEN} = \begin{bmatrix} I_r & 0 \\ 0 & A_f \end{bmatrix}, \quad \text{MAN} = \begin{bmatrix} A_s & 0 \\ 0 & I_{n-r} \end{bmatrix}. \quad (9)$$

The matrices M and N may also be used to decompose (4), yielding

$$MB = \begin{bmatrix} B_s \\ B_f \end{bmatrix}, \quad CN = [C_s \ C_f].$$

Referring to [14], we say that (4) is *slow controllable* if and only if

$$\text{rank } [\lambda E - A \ \ B] = n \quad (10)$$

for every $\lambda \in \mathbb{C}$ and *fast controllable* if and only if

$$\text{rank } [E \ \ B] = n. \quad (11)$$

The system is *controllable* if and only if both (10) and (11) hold. In addition, we say that (4) is *impulse controllable* if and only if

$$\text{Im } A_f + \text{Ker } A_f + \text{Im } B_f = \mathbb{R}^{n-r}. \quad (12)$$

(All four system properties can also be defined directly in terms of the solutions of the differential equation (4), but we find the linear algebraic characterizations more useful in the context of this paper.) Controllability and observability imply impulse controllability and impulse observability, respectively. The corresponding definitions for observability are dual to (10), (11), and (12) (see [14]). Since each of these definitions is invariant under the equivalence transformation (7), we may also consider the subsets $\mathcal{L}_{sc}$, $\mathcal{L}_{fc}$, $\mathcal{L}_c$, $\mathcal{L}_{ic}$, $\mathcal{L}_{so}$, $\mathcal{L}_{fo}$, $\mathcal{L}_o$, $\mathcal{L}_{io} \subset \mathcal{L}$ determined by (10), (11), (12), and their duals, as well as the controllable and observable systems $\mathcal{L}_{co} = \mathcal{L}_c \cap \mathcal{L}_o$. Various properties of these spaces are studied in [13]; for example, $\mathcal{L}_{fc}$ and $\mathcal{L}_{fo}$ are open, and $\mathcal{L}_{ic}$ and $\mathcal{L}_{io}$ are dense in $\mathcal{L}$.

Other important subsets of $\mathcal{L}$ are the singular subspace $\mathcal{L}^s$, consisting of all points $[E, A, B, C]$ with E singular, the regular subspace $\mathcal{L}^n = \mathcal{L} - \mathcal{L}^s$, and the subspace of unit index systems

$$\mathcal{L}_1 = \{\xi \in \mathcal{L} \ | \deg |sE - A| = \text{rank } E\}.$$

In [10] it is shown that $\mathcal{L}^n$ is open and dense in $\mathcal{L}$; from [13], $\mathcal{L}_{co} \cap \mathcal{L}^s$ is dense in $\mathcal{L}^s$.

Let $\mathfrak{D}$ be the set of all C^∞ functions φ: $\mathbb{R} \to \mathbb{R}$ with compact support and let $\mathfrak{D}_+$ be the space of distributions with support in $[0, \infty)$ (see [19]). To define convergence in $\mathfrak{D}_+$, we adopt the weak* topology: A sequence $f_k \in \mathfrak{D}_+$ is said to converge to f if $\langle f, \phi \rangle$ for every $\phi \in \mathfrak{D}$, where $\langle f_k, \phi \rangle$ denotes the functional f_k evaluated at the point ϕ.

Associated with each initial condition $x_0 \in \mathbb{R}^n$ and each piecewise continuous input u there exists a unique solution $\Psi_{x_0u}(\xi_k) \in \mathfrak{D}_+^n$ of the system ξ_k (see [17]). From linearity it follows that the solution can be decomposed into natural and forced response

$$\Psi_{x_0u}(\xi_k) = \Psi_{x_00}(\xi_k) + \Psi_{0u}(\xi_k).$$

Letting

$$\begin{bmatrix} B_{sk} \\ B_{fk} \end{bmatrix} = M_kB_k, \quad [C_{sk} \ \ C_{fk}] = C_kN_k, \quad \begin{bmatrix} x_{0sk} \\ x_{0fk} \end{bmatrix} = N_k^{-1}x_0$$

$$\begin{bmatrix} \psi^s_{x_0u}(\xi_k) \\ \psi^f_{x_0u}(\xi_k) \end{bmatrix} = N_k^{-1}\Psi_{x_0u}(\xi_k) \quad (13)$$

we have from [17] that

$$\Psi^s_{x_0u}(\xi_k) = \exp(A_{sk})x_0 + \exp(A_{sk}) * B_{sk}u \quad (14)$$

where $\exp(A) \in \mathfrak{D}_+^{n^2}$ is defined by

$$\exp(A)(t) = e^{tA}$$

and "$*$" denotes convolution. Each $\Psi^s_{x_0u}$ satisfies several properties of continuity. Indeed, convergence of A_{sk} guarantees uniform convergence of $\exp(A_{sk})$ on compact intervals and, hence, weak* convergence. Continuity of convolution with respect to both types of convergence assures that each sequence $\Psi^s_{x_0u}(\xi_k)$ converges weak* and uniformly on compact intervals whenever $\xi_k \xrightarrow{w} \xi$. Furthermore, since $\Psi_{x_0u}(\xi) = \Psi^s_{x_0u}(\xi)$ for $\xi \in \mathcal{L}^n$, Ψ_{x_0u} satisfies the same properties when restricted to $\mathcal{L}^n$.

To aid in writing a general expression for $\Psi^f_{x_0u}(\xi_k)$, we note that there exists a nonsingular matrix sequence T_k (not necessarily convergent) such that

$$T_k^{-1}A_{fk}T_k = \begin{bmatrix} \hat{A}_{fk} & 0 \\ 0 & \tilde{A}_{fk} \end{bmatrix} \quad (15)$$

where $\hat{A}_{fk}$ is nonsingular and $\tilde{A}_{fk}$ is nilpotent. Then from [17],

$$\Psi^f_{x_0u}(\xi_k) = T_k \begin{bmatrix} \exp(\hat{A}_{fk}^{-1}) & 0 \\ 0 & \sum_{l=1}^{q_k-1} \delta^l \tilde{A}^l_{fk} \end{bmatrix} T_k^{-1}x_{0fk} + T_k \begin{bmatrix} \exp(\hat{A}_{fk}^{-1}) * \hat{A}_{fk}^{-1}\hat{B}_{fk}u \\ -\sum_{l=0}^{q_k-1} \tilde{A}^l_{fk}\tilde{B}_{fk}u^l \end{bmatrix} \quad (16)$$

where

$$\begin{bmatrix} \bar{B}_{fk} \\ \tilde{B}_{fk} \end{bmatrix} = T_k^{-1} B_{fk}$$

and δ^i and u^i denote the ith distribution derivatives.

III. Problem Formulation

We are now in a position to precisely state the basic problem under consideration. For two reasons we are forced to select a rather abstract mathematical framework for our constructions. First, since perturbations leading to changes in order require the use of systems of the form (4) and since such systems can have impulsive solutions, the space $\mathfrak{D}_+$ of distributions and its associated weak* topology underlie all analyses. Second, it will be seen that in order to meaningfully incorporate the idea that small system perturbations should lead to only small changes in the entries of the matrices E, A, B, and C, it is necessary to identify systems according to the equivalence relation (7). Hence, we must work with the non-Euclidean system spaces $\mathfrak{L}(n, m, p)$.

We consider the problem of compensating the plant model (4) with a static system of the form

$$u = Ky + v \tag{17}$$

where K is a matrix and v is an external input. Let $\mathcal{P} = \mathfrak{L}(n, m, p)$ and $\mathcal{Q} = \mathbb{R}^{mp}$, and note that the closed-loop system takes the form

$$\mathcal{C}(\xi, K): \quad \begin{aligned} E\dot{x} &= (A + BKC)x + Bv \\ y &= Cx. \end{aligned} \tag{18}$$

In general, the system (18) may not exhibit existence and uniqueness of solutions or may respond to certain initial conditions with impulsive transients (see [15], [17]). Since we are only interested in choosing a compensator such that the resulting closed-loop system does not suffer from either of these defects, we restrict attention to

$$\mathcal{R} = \{(\xi, K) \in \mathcal{P} \times \mathcal{Q} \mid \deg |sE - (A + BKC)| = \operatorname{rank} E\}. \tag{19}$$

Adopting (19) is equivalent to assuming that (18) has unit index; hence, we may set $\mathfrak{J} = \mathfrak{L}_1(n, m, p)$. Note that the loop-closing map C is continuous with respect to manifold topology on $\mathfrak{L}$ (weak convergence); i.e., small changes in the entries of E, A, B, C, and K bring about only small changes in the closed-loop system matrices.

We say that a sequence ξ_k in $\mathcal{P}$ *converges weakly* to $\xi \in \mathcal{P}$ ($\xi_k \xrightarrow{s} \xi$) if $\Psi_{x_0u}(\xi_k) \to \Psi_{x_0u}(\xi)$ weak* for every x_0 and u and if $C_k \to C$. On the other hand, we say that a sequence $\xi_k \in \mathfrak{J}$ *converges strongly* in $\mathfrak{J}$ if each $\Psi_{x_0u}(\xi_k)$ converges uniformly on compact subintervals of $(0, \infty)$ and $C \to C_k$. Uniform convergence of solutions is meaningful for systems in $\mathfrak{J} = \mathfrak{L}_1$ only because unit index systems have no impulsive components in their solutions (see [17]). We have shown in [10] that strong convergence in $\mathcal{P}$ implies weak convergence in $\mathcal{P}$.[1] It is easy to verify that strong convergence in $\mathfrak{J}$ implies convergence of each $\Psi_{x_0u}(\xi_k)$ in the weak* sense (see [19]); hence, strong convergence implies weak convergence in $\mathfrak{J}$ as well.

Although strong convergence of ξ_k does not necessarily imply that the entries of the system matrices E_k, A_k, B_k, and C_k converge regardless of the representation (4) of ξ_k, such a strict requirement would not be particularly meaningful, since premultiplication of (4) by any nonsingular matrix produces an entirely equivalent representation. From [10] it does follow that strong convergence of ξ_k implies convergence of some representative sequence $(E_k, A_k, B_k, C_k) \in \Sigma(n, m, p)$. We are therefore justified in interpreting weak convergence in $\mathcal{R}$ and $\mathfrak{J}$ as convergence of system parameters and stating that a perturbation of a system which yields only small changes in system trajectories also results in only small variations in system parameters.

A plant–compensator pair $(\xi, K) \in \mathcal{R}$ is asymptotically robust (or K is a robust compensator for ξ) if $\mathcal{C}(\xi_k, K_k) \xrightarrow{s} \mathcal{C}(\xi, K)$ for every x_0, u whenever $\xi_k \xrightarrow{s} \xi$ and $K_k \to K$. This is equivalent to continuity of the loop-closing map $\mathcal{C}$ at (ξ, K) with respect to strong convergence in $\mathcal{R}$ and $\mathfrak{J}$. It is routine to verify that our definition of robustness can be couched in terms of topologies on $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{J}$ simply by imposing on each set the weakest topology that makes each map Ψ_{x_0u} continuous (see [20]). Our main problem of interest is to characterize the class of all robust plant–compensator pairs of $(\xi, K) \in \mathcal{R}$ for any given values of n, m, and p. Equivalently, we seek to describe the class of all compensators K which are robust with respect to a given plant model ξ.

IV. The Class of Asymptotically Robust Compensators

We begin by presenting a result which formalizes the intuitive idea that robustness can fail to hold only when the plant model (4) is singular. Note that when (4) is regular ($\xi \in \mathfrak{L}^n$), $\mathcal{R} = \mathbb{R}^{mp}$.

Proposition 3.1: If $\xi \in \mathfrak{L}^n$, every $K \in \mathbb{R}^{mp}$ is robust.

Proof: Choose K, x_0, u, and sequences $K_k \to K$ and $\xi_k \xrightarrow{s} \xi$. Then $C_k \to C$. Since $\mathfrak{L}^n$ is open in manifold topology, $\xi_k \in \mathfrak{L}^n$ for sufficiently large k. From [10], $\xi_k \xrightarrow{w} \xi$ so continuity of $\mathcal{C}$ with respect to weak convergence implies $\mathcal{C}(\xi_k, K_k) \xrightarrow{w} \mathcal{C}(\xi, K)$. Since each Ψ_{x_0u} is continuous on $\mathfrak{L}^n$ and $\mathcal{C}(\mathfrak{L}^n \times \mathbb{R}^{mp}) \subset \mathfrak{L}^n$, we have

$$\Psi_{x_0u}(\mathcal{C}(\xi_k, K_k)) \to \Psi_{x_0u}(\mathcal{C}(\xi, K)).$$

□

Before starting our main result on robustness, we need to consider one more algebraic system property of (4). We say that a system (4) is fast cyclic if, in the Weierstrass decomposition (8), the nilpotent matrix A_f is cyclic. If A_f is in Jordan form, fast cyclicity is equivalent to

$$A_f = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}.$$

Hence, from (9), a system (4) is fast cyclic if and only if rank $E = n - 1$. Note that fast cyclicity is independent of the choice of representation for ξ.

In order to prove that a certain algebraic condition on the compensator K is well defined, we next present a pair of lemmas. It will eventually be proven that this condition is necessary and sufficient for robustness.

Lemma 4.1: Let N and T be any $n \times n$ matrices with N nilpotent and having index q. Then Ker N is $N^{q-1}T$-invariant.

Proof: Since $N(N^{q-1}T) = N^qT = 0$, Im $N^{q-1}T \subset$ Ker N. Hence,

$$(N^{q-1}T) \operatorname{Ker} N \subset \operatorname{Ker} N.$$

□

Note that, if N is cyclic, Ker N is one-dimensional.

Lemma 4.2: If $\xi \in \mathfrak{L}^s$ is fast cyclic, impulse controllable, and impulse observable, and (A_f, B_f, C_f) is obtained from the

[1] Actually, it is shown in [10] that convergence of $\Psi_{x_0u}(\xi_k)$ for every x_0, u guarantees convergence of ξ_k in manifold topology when u ranges over $\mathfrak{D}_+^m$. It is easy to show, however, that the same result holds when u is restricted to be piecewise continuous.

Weierstrass decomposition of any representation of ξ, then

$$A_f^{n-r-1}(I_{n-r}+B_fKC_f)\ |\mathrm{Ker}\ A_f>0 \tag{20}$$

determines a nonempty open affine half-space in $\mathbb{R}^{mp}$ which is independent of the representation. (The vertical bar denotes the restriction of the linear operator to the subspace Ker A_f.)

Proof: In view of Lemma 4.1, (20) is well-defined. For the case $r = n - 1$, $A_f = 0$ and

$$A_f^{n-r-1}(I+B_fKC_f)=1+B_fKC_f. \tag{21}$$

For $r < n - 1$, choose a nonsingular T so that $T^{-1}A_fT$ is in Jordan form. Letting

$$\begin{bmatrix} b_1 \\ \vdots \\ b_{n-r} \end{bmatrix} = T^{-1}B_f,\ [c_1 \cdots c_{n-r}]=C_fT \tag{22}$$

we know from [14] that impulse controllability and impulse observability guarantee $b_{n-r} \neq 0$ and $c_1 \neq 0$. Also,

$$A_f^{n-r-1}(I_{n-r}+B_fKC_f)$$
$$=T\begin{bmatrix} b_{n-r}Kc_1 & \cdots & b_{n-r}Kc_{n-r-1} & 1+b_{n-r}Kc_{n-r} \\ 0 & \cdot\ \cdot\ \cdot & \cdot\ \cdot & 0 \end{bmatrix}T^{-1}.$$

Hence,

$$A_f^{n-r-1}(I_{n-r}+B_fKC_f)\ |\mathrm{Ker}\ A_f=b_{n-r}Kc_1. \tag{23}$$

Setting (21) and (23) positive determines nonempty open affine half-spaces.

From [11], (A_f, B_f, C_f) is unique up to similarity transformation for a given ξ. Clearly, similarity transformation does not alter (21), so the resulting half-space is unchanged. To see how (23) is affected by similarity transformation, note that (23) means

$$A_f^{n-r-1}(I_{n-r}+B_fKC_f)x=b_{n-r}Kc_1x$$

for any $x \in \mathrm{Ker}\ A_f$. Let $z = T^{-1}x$. Then

$$T^{-1}A_f^{n-r-1}(I_{n-r}+B_fKC_f)Tz=b_{n-r}Kc_1z$$

so

$$(T^{-1}A_fT)^{n-r-1}(I_{n-r}+(T^{-1}B_f)$$
$$\cdot K(C_fT))|(T^{-1}\ \mathrm{Ker}\ A_f)=b_{n-r}Kc_1.$$

But T^{-1} Ker A_f = Ker $(T^{-1}AT)$ so the resulting half-space is again unchanged. □

A final technical lemma is needed to prove our main robustness theorem.

Lemma 4.3: Let a_{ik}; $i = 0, \cdots, \mu$ be convergent sequences in $\mathbb{R}$ with a $a_{\mu k} \neq 0$ for every k, and let f_{ik}: $\mathbb{R} \to \mathbb{R}$; $i = 1, \cdots, \nu - 1$; $k = 1, 2, \cdots$ be continuous at the origin and satisfy $f_{ik}(0) = 0$, where $\nu > \mu$. Then there exists a sequence ϵ_k in $\mathbb{R}$ such that for each k:

1) $0 < |\epsilon_k| < 1/k$
2) $\mathrm{sgn}\ \epsilon_k = -\mathrm{sgn}\ a_{\mu k}$
3) the polynomial $\epsilon_k s^\nu + f_{\nu-1,k}(\epsilon_k)s^{\nu-1} + \cdots + f_{\mu+1,k}(\epsilon_k)s^{\mu+1} + (a_{\mu k} + f_{\mu k}(\epsilon_k))s^\mu + \cdots + (a_{1k} + f_{1k}(\epsilon_k))s + a_{0k}$ has at least one real root λ_k with $\lambda_k > k$.

Proof: Fix k, let $\alpha_j = -1/j\ \mathrm{sgn}\ a_{\mu k}$, and consider the sequence (in j)

$$p_j(s)=\alpha_js^\nu+f_{\nu-1,k}(\alpha_j)s^{\nu-1}+\cdots$$
$$+f_{\mu+1}(\alpha_j)s^{\mu+1}+(a_{\mu k}+f_{\mu k}(\alpha_j))s^\mu+\cdots$$
$$+(a_{1k}+f_{1k}(\alpha_j))s+a_{0k}. \tag{24}$$

From [12, Lemma 4.3], p_j can be factored as

$$p_j(s)=\varphi_j(s^\mu+b_{\mu-1,j}s^{\mu-1}+\cdots+b_{0j})\prod_{i=1}^{\nu-\mu}(\sigma_{ij}s-1) \tag{25}$$

where each $\sigma_{ij} \to 0$ and φ_j, b_{ij}; $i = 0, \cdots, \mu - 1$ all converge. Equating the coefficients in (24) and (25) of s^ν and s^μ yields

$$\alpha_j=\varphi_j\prod_{i=1}^{\nu-\mu}\sigma_{ij}$$

$$a_{\mu k}=(-1)^{\nu-\mu}\lim\varphi_j.$$

For sufficiently large j it follows that

$$\mathrm{sgn}\prod_{i=1}^{\nu-\mu}\sigma_{ij}=\mathrm{sgn}\ \alpha_j\ \mathrm{sgn}\ \varphi_j$$
$$=-\mathrm{sgn}\ a_{\mu k}\ \mathrm{sgn}\lim\varphi_j$$
$$=(-1)^{\nu-\mu+1}.$$

Hence, for each sufficiently large j there must exist an i such that

$$\sigma_{ij} \in \mathbb{R},\ \sigma_{ij}>0.$$

Since $\sigma_{ij} \to 0$, there exists $j > k$ such that $1/\sigma_{ij} > k$. Set $\lambda_k = 1/\sigma_{ij}$ and $\epsilon_k = \alpha_j$. □

Our main result, Theorem 4.4 completely characterizes the robust static compensator gains K.

Theorem 4.4: Let $\xi \in \mathfrak{L}^s$.

1) A robust $K \in \mathbb{R}^{pm}$ exists iff ξ is fast cyclic, impulse controllable, and impulse observable.

2) Under the conditions of part 1), K is robust iff

$$A_f^{n-r-1}(I+B_fKC_f)\ |\mathrm{Ker}\ A_f>0. \tag{26}$$

Proof:

i) (Necessary): Let r = ord ξ. We need only consider the case $r < n - 1$, since $r = n - 1$ implies ξ is fast cyclic, impulse controllable, and impulse observable (see [14]). Suppose $r < n - 1$ and choose a representation (E, A, B, C) for ξ. Invoke the Weierstrass decomposition (9), select a similarity transformation to put A_f in Jordan form, use the notation (22), and let

$$\begin{bmatrix} 0 & \gamma_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \gamma_{n-r-1} \\ & & & 0 \end{bmatrix}=T^{-1}A_fT. \tag{27}$$

(Each γ_i is either 0 or 1.) If ξ is not fast cyclic, impulse controllable, or impulse observable, then either $\gamma_i = 0$ for some i, $b_{n-r} = 0$, or $c_1 = 0$ (see [14]). Choose nonzero sequences $\gamma_{ik} \to \gamma_i$, $b_{n-r,k} \to b_{n-r}$, $c_{1k} \to c_1$, and $K_k \to K$ such that

$$b_{n-r,k}K_kc_{1k}\prod_{i=1}^{n-r-1}\gamma_{ik}<0 \tag{28}$$

for every k, and define

$$B_{fk}=T\begin{bmatrix} b_1 \\ \vdots \\ b_{n-r-1} \\ b_{n-r,k} \end{bmatrix},\quad C_{fk}=[c_{1k}c_2 \cdots c_{n-r}]T^{-1} \tag{29}$$

$$A_{fk}(x)=T\begin{bmatrix} x & \gamma_{1k} & & \\ & \ddots & \ddots & \\ & & \ddots & \gamma_{n-r-1,k} \\ & & & x \end{bmatrix}T^{-1}. \tag{30}$$

Now, we may uniquely define sequences a_{ik}; $i = 0, \cdots, n-1$ and polynomials p_{ik}; $i = 1, \cdots, n-1$, with $p_{ik}(0) = 0$, according to

$$x^{n-r}s^n+(a_{n-1,k}+p_{n-1,k}(x))s^{n-1}+\cdots+(a_{1k}+p_{1k}(x))s+a_{0k}$$
$$=\begin{vmatrix} sI-(A_s+B_sK_kC_s) & -B_sK_kC_{fk} \\ -B_{fk}K_kC_s & sA_{fk}(x)-(I+B_{fk}K_kC_{fk}) \end{vmatrix}.$$

By elementary matrix arguments,

$$a_{n-1,k}=(-1)^{n-r}b_{n-r,k}K_kc_{1k}\prod_{i=1}^{n-r-1}\gamma_{ik}. \tag{31}$$

Letting $f_{ik}(x) = p_{ik}(x^{1/(n-r)})$, $\nu = n$, and $\mu = n-1$, we may select a sequence ϵ_k satisfying the properties in Lemma 4.3 and define

$$\alpha_k=-\left(\frac{1}{\epsilon_k}\right)^{1/(n-r)}.$$

Since $\epsilon_k \to 0$ and $\operatorname{sgn}\epsilon_k = -\operatorname{sgn} a_{n-1,k}$, (28) and (31) guarantee that $\alpha_k \to +\infty$. If we set

$$E_k=M^{-1}\begin{bmatrix} I_{n-r} & 0 \\ 0 & A_{fk}\left(-\frac{1}{\alpha_k}\right)\end{bmatrix}N^{-1},\ A_k=A$$

$$B_k=M^{-1}\begin{bmatrix} B_s \\ B_{fk}\end{bmatrix},\ C_k=[C_s\ C_{fk}]N^{-1} \tag{32}$$

we have

$$\det(sE_k-(A_k+B_kK_kC_k))$$
$$=\beta(\epsilon_ks^n+(a_{n-1,k}+f_{n-1,k}(\epsilon_k))s^{n-1}+\cdots$$
$$+(a_{1k}+f_1(\epsilon_k))s+a_{0k}) \tag{33}$$

for some constant β. From Lemma 4.3, (33) has at least one real root $\lambda_k > k$ for every k. Since (33) is just the characteristic polynomial of the closed-loop system $\mathcal{C}(\xi, K)$, (15) shows that λ_k must be an eigenvalue of the closed-loop $\bar{A}_{fk}^{-1}$ for sufficiently large k. Thus, $\exp(\bar{A}_{fk}^{-1})$ cannot converge uniformly on compact subintervals of $(0, \infty)$, since this would imply uniform convergence of its eigenvalue $\exp(\lambda_k)$. Letting $u = 0$, it follows from (13) and (16) that $\Psi_{x_0u}(\mathcal{C}(\xi_k, K))$ does not converge for every x_0.

In order to prove that K is not robust, we have only left to show that $\Psi_{x_0u}(\xi_k) \to \Psi_{x_0u}(\xi)$ in the weak* sense, where $\xi_k = [E_k, A_k, B_k, C_k]$. To do so, we note that

$$\Psi^f_{x_0uk}=\exp\left(A_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}\right)x_0+A_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}$$
$$\cdot\exp\left(A_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}\right)*B_{fk}u$$

where

$$\exp\left(tA_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}\right)=\sum_{i=0}^{n-1}\frac{1}{i!}\frac{d^i}{ds^i}e^{t/s}\Big|_{s=-1/\alpha_k}\cdot\begin{bmatrix} 0 & \gamma_{1k} & & \\ & \ddots & \ddots & \\ & & \ddots & \gamma_{n-r-1,k} \\ & & & 0 \end{bmatrix}^i$$

and

$$A_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}\exp\left(tA_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}\right)$$
$$=\sum_{i=0}^{n-1}\frac{1}{i!}\frac{d^i}{ds^i}\frac{1}{s}e^{t/s}\Big|_{s=-1/\alpha_k}\cdot\begin{bmatrix} 0 & \gamma_{1k} & & \\ & \ddots & \ddots & \\ & & \ddots & \gamma_{n-r-1,k} \\ & & & 0 \end{bmatrix}^i*B_{fk}u.$$

Consider the matrix

$$\Sigma_k=\begin{bmatrix} -\frac{1}{\alpha_k} & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -\frac{1}{\alpha_k}\end{bmatrix}.$$

A routine calculation shows that the (i, j)th entry of $\Sigma_k^{-1}\exp(t\,\Sigma_k^{-1})$ is

$$\sigma_{ijk}=\frac{1}{(j-i)!}\frac{d^{j-i}}{ds^{j-i}}\left(\frac{1}{s}e^{t/s}\right)\Big|_{s=-1/\alpha_k},\qquad j\geq i.$$

It was shown in the proof of [10, Theorem 4] that

$$\Sigma_k^{-1}\exp(\Sigma_k^{-1})\to-\begin{bmatrix} 1 & \delta & \cdots & \delta^{n-r-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \delta \\ & & & 1\end{bmatrix}.$$

Therefore, $\sigma_{ijk} \to -\delta^{j-i}$ for each $j \geq i$, and $\Psi^f_{x_0uk} \to \Psi^f_{x_0u}$ for any x_0u. It follows from (13), weak* continuity of convolution, and continuity of $\Psi^s_{x_0u}$ that $\Psi_{x_0u}(\xi_k) \to \Psi_{x_0u}(\xi)$.

(Sufficient): Let ξ_k be any sequence in $\mathcal{L}$ such that $\Psi_{x_0u}(\xi_k) \to \Psi_{x_0u}(\xi)$ for every x_0u and with $C_k \to C$. Then, from [10], $\xi_k \to \xi$ and the decomposition (8) may be invoked. It follows that

$$|M_k|\,|sE_k-A_k|\,|N_k|=|sI_r-A_{sk}|\,|sA_{fk}-I_{n-r}|$$

for any convergent representation (E_k, A_k, B_k, C_k). Suppose $(-1)^{n-r}|A_{fk}| < 0$ for infinitely many k. Since $|A_{fk}|$ is just the product of the eigenvalues of A_{fk}, there exists a subsequence of A_{fk_j} with at least one real, positive eigenvalue for each k_j. It follows that $A_{fk_j}^{-1}$ has an eigenvalue $\lambda_j \to \infty$. Let $u = 0$ and observe that

$$\langle\Psi^f_{x_0u}(\xi_{kj}), \phi\rangle=\Gamma_{k_j}x_{0fk_j}$$

where

$$\Gamma_{k_j}=\int_0^\infty \phi(t)e(tA_{fk_j}')\,dt.$$

A function $\phi \in \mathfrak{D}$ can always be chosen such that an eigenvalue of Γ_{k_j} satisfies

$$\int_0^\infty \exp(\lambda\cdot t)\phi(t)\,dt\to\infty.$$

Hence, $\|\Gamma_{k_j}\|\to\infty$. It follows that $\Gamma_k x_{0fk}$ is unbounded for an appropriate choice of x_0 and that $\Psi^f_{x_0u}(\xi_k)$ and [from (13)] $\Psi_{x_0u}(\xi_k)$ are not convergent. This contradiction leads us to conclude that $(-1)^{n-r}|A_{fk}|\geq 0$ for sufficiently large k.

Appealing to the notation of (22) and (27), we have b_{n-r}, $c_1\neq 0$ and, if $r<n-1$, $\gamma_1=\cdots=\gamma_{n-r-1}=1$. Choose K to satisfy the condition (26). For $r=n-1$, (21) indicates that $1+b_1Kc_1>0$; for $r<n-1$, (23) implies that $b_{n-r}Kc_1>0$. Defining

$$\Delta_k(s)=|sE_k-(A_k+B_kK_kC_k)|$$

we have

$$\begin{aligned}|M_k|\Delta_k(s)|N_k|&=\begin{vmatrix} sI_r-(A_{sk}+B_{sk}K_kC_{sk}) & -B_{sk}K_kC_{fk}\\ -B_{fk}K_kC_{sk} & sA_{fk}-(I_{n-r}+B_{fk}K_kC_{fk})\end{vmatrix}\\ &=|A_{fk}|s^n+(\alpha_k-|A_{fk}|\,tr(A_{sk}+B_{sk}K_kC_{sk}))s^{n-1}+\cdots\end{aligned}\tag{34}$$

where α_k is defined by

$$|sA_{fk}-(I_{n-r}+B_{fk}K_kC_{fk})|=|A_{fk}|s^{n-r}+\alpha_ks^{n-r-1}+\cdots.$$

From elementary matrix arguments we have, for $r<n-1$,

$$|sA_f-(I_{n-r}+B_fKC_f)|=(-1)^{n-r}b_{n-r}Kc_1s^{n-r-1}+\cdots$$

so

$$\lim\alpha_k=\begin{cases}-(1+b_1Kc_1) & \text{if } n-r=1\\ (-1)^{n-r}b_{n-r}Kc_1 & \text{if } n-r>1.\end{cases}\tag{35}$$

From our choice of K it follows that the closed-loop system $\mathcal{C}(\xi, K)$ exhibits no impulsive behavior in its natural response, i.e., ord $\mathcal{C}(\xi,K)=\text{rank }E=n-1$. Hence, from [12, Lemma 4.3] we know that

$$|M_k|\Delta_k(s)|N_k|=\zeta_k(\sigma_ks-1)\prod_{l=1}^{n-1}(s-\lambda_{lk})\tag{36}$$

where φ_k, σ_k, and λ_k all converge and $\lim\sigma_k=0$. Matching coefficients in (34) and (36) yields

$$\varphi_k\sigma_k=|A_{fk}|\tag{37}$$

$$\lim\alpha_k=-\lim\varphi_k.\tag{38}$$

From (35) and our choice of K, $(-1)^{n-r}\lim\alpha_k<0$. Hence, from (38),

$$(-1)^{n-r}\lim\varphi_k=(-1)^{n-r+1}\lim\alpha_k<0.$$

Thus, $(-1)^{n-r}\varphi_k<0$ for sufficiently large k, and

$$\sigma_k=\frac{(-1)^{n-r}|A_{fk}|}{(-1)^{n-r}\varphi_k}\leq 0.$$

Applying the decomposition (8) to the closed-loop system yields nonsingular transformations $\bar M_k$ and $\bar N_k$ such that

$$\bar M_kE_k\bar A_k=\begin{bmatrix} I_{n-1} & 0\\ 0 & \sigma_k\end{bmatrix},\ \bar M_k(A_k+B_kK_kC_k)\bar N_k=\begin{bmatrix}\bar A_{sk} & 0\\ 0 & 1\end{bmatrix}$$

$$\bar M_kB_k=\begin{bmatrix}\bar B_{sk}\\ \bar B_{fk}\end{bmatrix},\ C_k\bar N_k=[\bar C_{sk}\ \ \bar C_{fk}]$$

where all sequences converge. The decomposition (13) may also be applied to the closed-loop system yielding

$$\Psi_{x_0u}(\mathcal{C}(\xi_k))=\bar N_k\begin{bmatrix}\Psi^s_{x_0u}(\xi_k)\\ \Psi^f_{x_0u}(\xi_k)\end{bmatrix}.$$

From Lemma 4.3, $\Psi^s_{x_0u}(\xi_k)\to\Psi^s_{x_0u}(\xi)$, and

$$\Psi^f_{x_0u}(\xi_k)=\begin{cases}\exp\left(\dfrac{1}{\sigma_k}\right)x_{0fk}+\dfrac{1}{\sigma_k}\exp\left(\dfrac{1}{\sigma_k}\right)*\bar B_{fk}u & \text{if } \sigma_k<0\\ -\bar B_{fk}u\end{cases}$$

so, as in the necessity proof of part 1), $\Psi^f_{x_0u}(\xi_k)\to\Psi^f_{x_0u}(\xi)$ for any x_0, u. Hence, $\Psi_{x_0u}(\mathcal{C}(\xi_k))\to\Psi_{x_0u}(\mathcal{C}(\xi))$ and K is robust.

2) (Sufficient): This part has already been treated in the Sufficiency section of 1).

(Necessary): Invoke the Weierstrass decomposition (9). If (26) fails and $r=n-1$, we have $1+B_fKC_f\leq 0$ so $B_f\neq 0$ and $C_f\neq 0$; hence, there exists a sequence $K_k\to K$ such that

$$1+B_fK_kC_f<0\tag{39}$$

for every k. Now define $a_{1k},\cdots,a_{n-1,k};\ \beta_{1k},\cdots,\beta_{n-1,k}$ according to

$$xs^n+(a_{n-1,k}+\beta_{n-1,k}x)s^{n-1}+\cdots+(a_{1k}+\beta_{1k}x)s+a_{0k}=\begin{vmatrix} sI_{n-1}-(A_s+B_sK_kC_s) & -B_sK_kC_f\\ -B_fK_kC_s & xs-(1+B_fK_kC_f)\end{vmatrix}.$$

Then

$$a_{n-1,k}=-(1+B_fK_kC_f).\tag{40}$$

Letting $f_{ik}(x)=\beta_{ik}x$, we can find a sequence ϵ_k satisfying the properties in Lemma 4.3; define $\alpha_{nk}=-1/\epsilon_k$. Since $\epsilon_k\to 0$ and $\text{sgn }\epsilon_k=-\text{sgn }a_{n-1,k}$, (39) and (40) guarantee that $\alpha_k\to+\infty$. If we set

$$E_k=M^{-1}\begin{bmatrix} I_{n-1} & 0\\ 0 & -\dfrac{1}{\alpha_k}\end{bmatrix}N^{-1},\ A_k=A$$

$$B_k=B,\ C_k=C$$

we have that $\det(sE_k-(A_k+B_kK_kC_k))$ has at least one real root $\lambda_k>k$ for each k. As in the sufficiency proof of part 1), $\Psi_{x_0u}(\mathcal{C}(\xi_k,K_k))$ does not converge for some x_0, u. Since

$$\Psi^f_{x_0u}(\xi_k)=\exp(-\alpha_k)x_0-\alpha_k\exp(-\alpha_k)*B_{fk}u\tag{41}$$

in the open-loop system, $\Psi_{x_0u}(\xi_k)\to\Psi_{x_0u}(\xi)$ and K is not robust.

If (26) fails and $r<n-1$, we may adopt the notation (29) and (30) and observe that $b_{n-r}Kc_1\leq 0$. Since fast cyclicity, impulse controllability, and impulse observability guarantee that $b_{n-r}\neq 0$ and $c_1\neq 0$, a sequence $K_k\to K$ may be chosen so that $b_{n-r}K_kc_1<0$ for every K. The remaining arguments are the same as in the necessity part of 1) with $b_{n-r,k}=b_{n-r}$, $c_{1k}=c_1$, and $\gamma_{ik}=1$. □

Theorem 4.4 is somewhat pessimistic in that, in the strictest

theoretical sense, robustness can only be guaranteed when at most one degree of singularity is present in the plant (4) (rank $E = n - 1$). In physical terms this can be interpreted as meaning that a static compensator can handle only a first-order unmodeled dynamic element. In our opinion, this indicates that some basic assumptions which are as yet not well understood are conventionally placed on system models in engineering practice.

For a mathematical explanation of how nonrobust compensators may fail to stabilize a system, consider the matrix condition in part 2) of Theorem 4.4. This condition determines an open affine half-space in the set $\mathfrak{R}^{mp}$ of compensation gains K. Examination of the proof of Theorem 4.4 reveals that, for systems which are fast cyclic, impulse controllable, and impulse observable, a static compensator results in positive feedback either for all admissible perturbations simultaneously or for none at all. The half-space of robust feedback gains is simply the set of all K with the appropriate sign to guarantee negative feedback for all perturbations of the system (4). The system (6) illustrates this point. The robust gains are simply those satisfying $K < 1$. On the other hand, part 1) of Theorem 4.4 maintains that unless the plant is fast cyclic, impulse controllable, and impulse observable, the class of admissible perturbations is so broad that any compensator results in positive feedback with respect to some perturbation; hence, no compensator is robust. This is illustrated by (2) and (3).

Another important point to note at this stage is that, although all definitions and technical arguments until now have been couched in terms of sequences, each statement applies equally well to nets in the various topological spaces. This observation is important, since the space of distributions $\mathfrak{D}_+$ does not satisfy the first axiom of countability (see [20]).

To conclude this section we compare our results to those of [5]. Specifically, [5, Theorem 1] shows that, for any system

$$\dot{x} = Ax + Bu$$
$$y = Cx \tag{42}$$

and any compensation matrix K, there exists a singular perturbation of (42) of the form (1) which destabilizes the closed-loop system. (The result of [5] is somewhat more general in that it applies to all dynamic compensators which are proper but not strictly proper.) According to Theorem 4.4, if we take such a perturbation and set $\epsilon = 0$, we obtain a nominal system

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$y = [C_1 \;\; C_2] x \tag{43}$$

which must either fail to be fast cyclic, impulse controllable, or impulse observable. For example, setting $\epsilon = 0$ in (3) yields a system of the form (43) which can be shown to be not fast cyclic. While the result of [5] illustrates that a specialized class of parasitics can lead to closed-loop destabilization, our results characterize the same phenomenon but in the context of a broader class of perturbations and a larger family of nominal systems. For example, our Theorem 4.4 applies to systems of the form (43) with A_{22} singular (as long as $|sE - A| \not\equiv 0$ is satisfied), while [5] considers only the case of A_{22} nonsingular. Our result also shows when destabilization can occur as a result of perturbations to a given order; the perturbed order required to destabilize the closed-loop system in [5] is not specified.

V. Genericity

We now consider the class of systems (4) for which there exists a robust compensator K. The sets of impulse controllable and impulse observable systems were shown in [13] to be dense in the system space $\mathfrak{L}$. The next result characterizes those systems which are also fast cyclic.

Proposition:

1) ξ is fast cyclic iff $\xi \in \mathfrak{L}^n \cup \mathfrak{L}^{n-1}$.
2) $(\mathfrak{L}^n \cup \mathfrak{L}^{n-1}) \cap \mathfrak{L}_{ic} \cap \mathfrak{L}_{io}$ is open in $\mathfrak{L}$.
3) $\mathfrak{L}^{n-1} \cap \mathfrak{L}_{ic} \cap \mathfrak{L}_{io}$ is dense in $\mathfrak{L}^s$.

Proof:

1) Let (E, A, B, C) be any representative of ξ. If $r = \text{ord}\, \xi$, the Weierstrass decomposition (9) shows that rank $E = r +$ rank A_f. But ξ is fast cyclic if and only if either $r = n$ or rank $A_f = n - r - 1$. Hence, ξ is fast cyclic iff rank $E = n$ or rank $E = n - 1$.

2) Let

$$\xi = [E, A, B, C] \in \Omega = (\mathfrak{L}^n \cup \mathfrak{L}^{n-1}) \cap \mathfrak{L}_{ic} \cap \mathfrak{L}_{io}$$

and apply the decomposition (9). Then A_f is cyclic, and Ker $A_f \subset$ Im A_f. Since ξ is impulse controllable and impulse observable.

$$\text{Im}\, A_f + \text{Im}\, B_f = \text{Im}\, A_f + \text{Ker}\, A_f + \text{Im}\, B_f = \mathfrak{R}^{n-r}$$

$$\text{Ker}\, A_f \cap \text{Ker}\, C_f = \text{Ker}\, A_f \cap \text{Im}\, A_f \cap \text{Ker}\, C_f = 0$$

so $\xi \in \mathfrak{L}_{fc} \cap \mathfrak{L}_{fo}$ (see [14]). It follows that

$$\Omega = (\mathfrak{L}^n \cap \mathfrak{L}^{n-1}) \cap \mathfrak{L}_{fc} \cup \mathfrak{L}_{fo}.$$

We know from [13] that $\mathfrak{L}_{fc}$ and $\mathfrak{L}_{fo}$ are both open, so Ω is the finite intersection of open sets.

3) It was shown in [13] that $\mathfrak{L}_{co} \cap \mathfrak{L}^{n-1}$ is dense in $\mathfrak{L}^s$. Our result follows immediately, since $\mathfrak{L}_{ic} \cap \mathfrak{L}_{io} \supset \mathfrak{L}_{co}$. □

Note that part 3) is stated in terms of the singular subspace $\mathfrak{L}^s$. Since every point in the regular subspace $\mathfrak{L}^n$ is necessarily fast cyclic, impulse controllable, and impulse observable (see [14]) and since $\mathfrak{L}^n$ is dense in $\mathfrak{L}$, density of $(\mathfrak{L}^n \cup \mathfrak{L}^{n-1}) \cap \mathfrak{L}_{ic} \cap \mathfrak{L}_{io}$ in $\mathfrak{L}$ is trivial. Part 3) is a much stronger result.

VI. Discussion and Conclusions

In this section we discuss some of the implications of our theory and use these to suggest further research. Theorem 4.5 shows that a generic class of systems can be robustly compensated using static compensators K. This does not mean, however, that the complement of the open and dense subset $(\mathfrak{L}^n \cup \mathfrak{L}^{n-1}) \cap \mathfrak{L}_{ic} \cap \mathfrak{L}_{io}$ does not contain interesting systems. On the contrary, it is easy to show that all systems of the form (4) with $r < n - 1$ and A_{22} nonsingular lie outside the generic class described by Theorem 4.5. Another interesting observation is that even a system which does lie in the generic set can be trivially augmented so that it sits outside the generic set in a higher dimensional system space. For example, the dimension of (4) may be increased simply by defining a new (scalar-valued) state variable $z = 0$ and noting that

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$
$$y = [C \;\; 0] \begin{bmatrix} x \\ z \end{bmatrix}. \tag{44}$$

System (44) is a member of $\mathfrak{L}(n+1, m, p)$. It is easy to show that (44) is not fast cyclic and, hence, cannot be robustly compensated. The latter point can be countered by arguing that only variables of interest should be included in a well-devised state-space model; therefore, the variable z would never be present.

There are at least a couple of avenues of research which might eventually resolve these issues. Dynamic compensation is still relatively unexplored in the context of singular perturbations. One promising result is [5, Theorem 2] which suggests that, when parasitics are present, strictly proper compensators are more robust than nonstrictly proper ones. Since [5] treats only the single time-scale case, more work needs to be done to see whether this

result stands up to a larger class of perturbations. As pointed out in Section I, the issue of which class of perturbations is meaningful in a given system analysis is of fundamental importance. Our main results can in fact be proven under a somewhat more general definition of system perturbation than the one provided here (strong convergence). However, preliminary work suggests that even such a generalization might be too restrictive to allow a coherent robustness theory to be developed. We intend to explore these issues more fully in the future.

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J. Daniel Cobb (M'82) was born in Chicago, IL, in 1953. He received the B.S. degree in electrical engineering from the Illinois Institute of Technology, Chicago, in 1975 and the M.S. and Ph.D. degrees in electrical engineering from the University of Illinois, Urbana, in 1977 and 1980, respectively.

From 1977 to 1980 he was a Research Assistant at the Coordinated Science Laboratory, University of Illinois, Urbana. He was then Visiting Assistant Professor in the Department of Electrical Engineering, University of Toronto, Toronto, Ontario, Canada. At present he is with the Department of Electrical and Computer Engineering, University of Wisconsin, Madison.

Dr. Cobb is presently an Associate Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL. He is a member of the American Mathematical Society, the Society for Industrial and Applied Mathematics, and the IEEE Control Systems Society.

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THE MINIMAL DIMENSION OF STABLE FACES REQUIRED TO GUARANTEE STABILITY OF A MATRIX POLYTOPE[1]

J. Daniel Cobb
Christopher L. DeMarco

Department of Electrical and Computer Engineering
University of Wisconsin-Madison
1415 Johnson Drive
Madison, WI 53706-1691

Abstract

We consider the problem of determining whether each point in a polytope of $n \times n$ matrices is stable. Our approach is to check stability of certain faces of the polytope. For $n \geq 3$, we show that stability of each point in every $(2n-4)$-dimensional face guarantees stability of the entire polytope. Furthermore, we prove that, for any $k \leq n^2$, there exists a k-dimensional polytope containing a strictly unstable point and such that all its subpolytopes of dimension $\min\{k-1, 2n-5\}$ are stable.

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1. Background and Introduction

In this paper we consider the problem of ascertaining whether certain subsets of $\mathbb{R}^{n\times n}$ consist entirely of stable matrices. (Here we take stability of a matrix to mean that all its eigenvalues are in the open left half plane.) First we need some definitions. A <u>(convex) polytope</u> $\mathcal{P}$ in a vector space V is the convex hull of any nonempty finite subset of V. The <u>dimension</u> of $\mathcal{P}$ is the dimension of the affine hull aff($\mathcal{P}$) of $\mathcal{P}$. The <u>relative boundary</u> of $\mathcal{P}$ is the boundary of $\mathcal{P}$ as a subset of the topological space aff($\mathcal{P}$). A <u>face</u> of $\mathcal{P}$ is any set of the form $\Pi\cap\mathcal{P}$, where Π is a supporting hyperplane of $\mathcal{P}$. A <u>vertex</u> of $\mathcal{P}$ is a 0-dimensional face. An <u>edge</u> of $\mathcal{P}$ is a 1-dimensional face. A <u>subpolytope</u> of $\mathcal{P}$ is the convex hull of any set of vertices of $\mathcal{P}$. Finally, a <u>k-dimensional half-plane</u> in V is any nonempty set of the form $\mathcal{H}=R\cap S$, where R is a closed half-space, S is a k-dimensional affine subspace, and $S\not\subset R$. (Note that this implies that the affine hull of $\mathcal{H}$ is simply S.)

In the robust control literature, considerable interest has been generated by the problem of determining whether stability of a polytope in either $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$ can be guaranteed simply by checking stability of low-dimensional faces. (Stability of a vector $x\in\mathbb{R}^n$ means simply that the polynomial $s^n+x_ns^{n-1}+\ldots+x_1$ is Hurwitz.) We first note that the cases n=0 and n=1 are trivial; stability of the vertices always guarantees stability of the polytope. Several recent papers consider the case $n\geq2$. For example, polynomial polytopes of a particularly simple structure ("interval polynomials") were addressed by Kharitonov [1]; he showed that only four specially constructed vertices need be checked. A more recent result of Bartlett, Hollot, and Lin [2] demonstrates that, for an arbitrary polynomial polytope checking all edges is sufficient to guarantee stability of $\mathcal{P}$. With respect to polytopes in $\mathbb{R}^{n\times n}$, Fu and Barmish [3] have shown that stability of

all 1-dimensional subpolytopes is insufficient to guarantee stability of $\mathcal{P}$. DeMarco [4] has shown that, for $n \geq 3$, $(n-2)$-dimensional faces are insufficient, but $2n$-dimensional faces are sufficient.

In this paper we refine the bounds of [4] and show that stability of all m-dimensional faces is sufficient to guarantee stability of $\mathcal{P}$, where

$$m(n) = \begin{cases} 1, & n=2 \\ 2n-4, & n>2 \end{cases}$$

Furthermore, we show that for any n and $k \leq n^2$ there exists a polytope of dimension k, containing a strictly unstable point (a matrix with an eigenvalue λ satisfying $\mathrm{Re}\,\lambda > 0$), and such that all its $\min(k-1, m-1)$-dimensional subpolytopes are stable; hence, in this sense, m is minimal.

2. Sufficiency of m

Throughout our analysis, we will make extensive use of the fact that any affine, one-to-one map $f: \mathbb{R}^k \longrightarrow \mathbb{R}^{n^2}$ determines an affine isomorphism between $\mathbb{R}^k$ and $f(\mathbb{R}^k)$. Among other things, this implies that, for any polytope $\mathcal{P} \subset \mathbb{R}^k$, $f(\mathcal{P})$ is also a polytope of the same dimension as $\mathcal{P}$; furthermore, f sets up a one-to-one correspondence between q-dimensional faces of $\mathcal{P}$ and q-dimensional faces of $f(\mathcal{P})$. In addition, f maps each k-dimensional half-plane in $\mathbb{R}^k$ into another k-dimensional half-plane (e.g., see [5]). Finally, we note that every polytope is compact and that the set $\{x \in \mathbb{R}^k \,|\, \|x\|_\infty \leq 1\}$ is a polytope whose q-dimensional faces are generated by fixing $k-q$ entries of x at either ± 1 and letting the remaining q entries vary independently over $[-1, 1]$.

With these observations in mind, we prove a result characterizing the affine structure of the set of unstable points in $\mathbb{R}^{n \times n}$.

Lemma 2.1 For each unstable $A\in\mathbb{R}^{n\times n}$ there exists an (n^2-m) dimensional half-plane $\mathcal{H}\subset\mathbb{R}^{n\times n}$ such that 1) $A\in\mathcal{H}$ and 2) $B\in\mathcal{H}$ implies B is unstable.

Proof Case I -- A has a real eigenvalue $\lambda_0\geq 0$: Let $T=[v\ W]$, where v is an eigenvector corresponding to λ_0 and W is chosen to make T nonsingular. Clearly, the map $f:\mathbb{R}^{n^2-n+1}\longrightarrow\mathbb{R}^{n\times n}$ determined by

$$f(\lambda,y,Z)=T\begin{bmatrix}\lambda & y\\ 0 & Z\end{bmatrix}T^{-1}$$

is affine and one-to-one. Let $\bar{\mathcal{H}}$ be the (n^2-n+1)-dimensional half-plane

$$\bar{\mathcal{H}}=\{f(\lambda,y,Z)\,\big|\,\lambda\geq\lambda_0,\ y\in\mathbb{R}^{1\times n-1},\ Z\in\mathbb{R}^{n-1\times n-1}\}$$

Then $A\in\bar{\mathcal{H}}$ and every matrix in $\bar{\mathcal{H}}$ is unstable. Since $n^2-n+1\geq n^2-m$, we need only select any (n^2-m)-dimensional half-plane $\mathcal{H}$ satisfying $A\in\mathcal{H}\subset\bar{\mathcal{H}}$.

Case II -- A has a complex eigenvalue pair $\alpha_0\pm i\beta_0$ with $\alpha_0>0$: Let $T=[v\ w\ X]$, where $v+iw$ is an eigenvector corresponding to $\alpha_0+i\beta_0$ and X is chosen to make T nonsingular. Let $\bar{\mathcal{H}}$ be the (n^2-2n+4)-dimensional half-plane

$$\bar{\mathcal{H}}=\{T\begin{bmatrix}U & Y\\ 0 & Z\end{bmatrix}T^{-1}\,\big|\,\mathrm{tr}\,U\geq 2\alpha_0,\ Y\in\mathbb{R}^{2\times n-2},\ Z\in\mathbb{R}^{n-2\times n-2}\}$$

($\mathrm{tr}\,U\geq 2\alpha_0$ describes a 4-dimensional half-plane, since $\mathrm{tr}\,U=\langle U,I\rangle$.) $\bar{\mathcal{H}}$ contains only unstable points, since $\mathrm{tr}\,U\geq 2\alpha_0$ implies U has at least one eigenvalue λ with $\mathrm{Re}\,\lambda\geq\alpha_0$. Also, $A\in\bar{\mathcal{H}}$, since our choice of T guarantees that A has

$$U=\begin{bmatrix} \alpha_0 & \beta_0 \\ -\beta_0 & \alpha_0 \end{bmatrix}$$

Finally, $n^2-2n+4 \geq n^2-m$, so the desired $\mathcal{H} \subset \bar{\mathcal{H}}$ exists.

Next we prove an easy result concerning the intersection of affine sets.

<u>Lemma 2.2</u> Let V be a p-dimensional Euclidean space, $\mathcal{H} \subset V$ a k-dimensional half-plane, and Γ a q-dimensional affine subspace with $k+q>p$. Consider any vector $x_0 \in \mathcal{H} \cap \Gamma$. There exists a $(k+q-p)$-dimensional half-plane $\bar{\mathcal{H}}$ such that $x_0 \in \bar{\mathcal{H}} \subset \mathcal{H} \cap \Gamma$.

<u>Proof</u> By definition, $\mathcal{H}=R \cap S$, where R is a closed half-space and S is a k-dimensional affine subspace satisfying $S \not\subset R$. There exists an affine subspace $\bar{S} \subseteq S \cap \Gamma$ with $\dim \bar{S}=k+q-p$ and $x_0 \in \bar{S}$. If $\bar{S} \subset R$, let $\bar{\mathcal{H}} \subset R \cap \bar{S}$ be any $(k+q-p)$-dimensional half-space containing x_0. Then $\bar{\mathcal{H}} \subset R \cap S \cap \Gamma = \mathcal{H} \cap \Gamma$. If $\bar{S} \not\subset R$, let $\bar{\mathcal{H}}=R \cap \bar{S}$. Then $x_0 \in \bar{\mathcal{H}}$, since $x_0 \in \mathcal{H} \cap \Gamma \subset R$. Also, $\dim \bar{\mathcal{H}}=k+q-p$, since $\bar{\mathcal{H}}$ is nonempty. □

We are now in a position to prove our first main result.

<u>Theorem 2.3</u> Stability of every matrix in every m-dimensional face of $\mathcal{P}$ guarantees stability of every matrix in $\mathcal{P}$.

<u>Proof</u> Our arguments here are similar to those used in [2,Lemma 1]. Suppose $\mathcal{P}_k$ is an unstable polytope of dimension $k>m$. Then there exists an unstable matrix $A_1 \in \mathcal{P}_k$. From Lemma 2.1, there is a (n^2-m)-dimensional half plane $\mathcal{H}_1$, consisting entirely of unstable points and containing A_1. Since $\mathcal{H}_1$ is unbounded, there exists an $A_2 \in \mathcal{H}_1$ lying on the boundary of $\mathcal{P}_k$ and, hence, in

one of its $(k-1)$-dimensional faces $\mathcal{P}_{k-1}$. From Lemma 2.2, the intersection $\mathcal{H}_1 \cap \mathrm{aff}(\mathcal{P}_{k-1})$ contains a $(k-m-1)$-dimensional half-plane $\mathcal{H}_2$ such that $A_2 \in \mathcal{H}_2$. Proceeding inductively, we find that there exists an m-dimensional face $\mathcal{P}_m$ and a point $A_{k-m} \in \mathcal{P}_m$ such that A_{k-m} is unstable.

3. Minimality of m

Our next task is to show that m is the smallest integer such that stability of all m-dimensional faces of $\mathcal{P}$ guarantees stability of $\mathcal{P}$.

Theorem 3.1 For each integer $n \geq 2$ there exists an m-dimensional polytope $\mathcal{P} \subset \mathbb{R}^{n \times n}$ containing an unstable point and such that all its $(m-1)$-dimensional faces are stable.

Proof Case I -- n=2: Consider the affine, one-to-one map

$$f(x)=\begin{bmatrix} 0 & x \\ -x & -1 \end{bmatrix}$$

and the corresponding 1-dimensional polytope $\mathcal{P}=\{f(x) \,\big|\, |x| \leq 1\}$. Each matrix in $\mathcal{P}$ has characteristic polynomial $\Delta(s)=s^2+s+x^2$; hence, each vertex of $\mathcal{P}$ $(x=\pm 1)$ is stable, but the point corresponding to x=0 is unstable.

Case II -- n=3: The 2-dimensional polytope

$$\mathcal{P}=\left\{ \begin{bmatrix} 0 & 1 & -x \\ -1 & 0 & -y \\ x & y & -1 \end{bmatrix} \,\middle|\, \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_\infty \leq 1 \right\}$$

has characteristic polynomial $\Delta(s)=s^3+s^2+(1+x^2+y^2)s+1$. Each coefficient of

$\Delta(s)$ is positive, and the corresponding 2×2 Hurwitz matrix has its leading principal second-order minor equal to $M_2(x,y)=x^2+y^2$. Thus, each edge is stable, but the matrix corresponding to x=y=0 is unstable.

Case III -- n≥4: Consider the (2n-4)-dimensional polytope

$$\mathcal{P}=\left\{\begin{bmatrix} 0 & 1 & -x^T \\ -1 & 0 & -y^T \\ x & y & -I \end{bmatrix} \;\middle|\; \left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|_\infty \le 1\right\}.$$

A routine calculation shows that $\mathcal{P}$ has characteristic polynomial $p(s)=(s+1)^{n-4}\Delta(s)$, where

$$\Delta(s)=s^4+2s^3+(2+x^Tx+y^Ty)s^2+(2+x^Tx+y^Ty)s+1+x^Txy^Ty-(x^Ty)^2$$

From the Schwartz inequality, it is clear that all coefficients of Δ are strictly positive. The corresponding 4×4 Hurwitz matrix has its leading principal third-order minor equal to

$$M_3(x,y)=4x^Tx+4y^Ty+4(x^Ty)^2+(x^Tx-y^Ty)^2$$

Clearly, $M_3 \ge 0$ with equality if and only if x=y=0. Thus the (2n-5)-dimensional faces of $\mathcal{P}$ are stable, but the point corresponding to x=y=0 is unstable.

It is interesting to note that Theorem 3.1 also implies that the half-planes considered in Lemma 2.1 are maximal in the sense that there exists an unstable matrix A in $\mathbb{R}^{n\times n}$ such that every half-plane of dimension greater than n^2-m containing A must also contain a stable matrix. Indeed, if this

were not the case, the arguments in Theorem 2.3 could be used to prove that m is not minimal.

4. A Stronger Version of the Minimality Theorem

The construction in the proof of Theorem 3.1 is weak in three respects: 1) The polytope $\mathcal{P}$ contains only a single marginally unstable matrix (i.e. a matrix having all eigenvalues λ satisfying $\mathrm{Re}\,\lambda \le 0$ and at least one with $\mathrm{Re}\,\lambda = 0$). 2) The construction yields only a polytope of dimension m. 3) Arbitrary subpolytopes are not considered; thus it is not clear that checking all subpolytopes of dimension, say, m-1 would not guarantee stability. The minimality proof would be more convincing if it could be extended to give a family of polytopes, each 1) containing a strictly unstable point (and, hence, infinitely many unstable points), 2) having arbitrary dimension k, and 3) having all min{k-1,m-1}-dimensional subpolytopes stable.

Theorem 4.2 shows that such improvements over Theorem 3.1 can be made. The proof requires a simple lemma. For any normed linear space V, subset $\Omega \subset V$, and point $\gamma \in V$ consider the distance function

$$d(\gamma,\Omega)=\inf_{\omega\in\Omega}\|\gamma-\omega\|$$

Let conv(Ω) denote the convex hull of Ω.

Lemma 4.1 Suppose $\Omega \subset V$ is convex, $\varepsilon > 0$, and $\Gamma \subset V$ is any set such that $d(\gamma, \Omega) < \varepsilon$ for every $\gamma \in \Gamma$. Then $d(\eta, \Omega) < \varepsilon$ for every $\eta \in \mathrm{conv}(\Omega \cup \Gamma)$.

Proof Each $\eta \in \mathrm{conv}(\Omega \cup \Gamma)$ is of the form $\eta = \alpha\eta_1 + (1-\alpha)\eta_2$, where $\alpha \in [0,1]$ and $\eta_1, \eta_2 \in \Omega \cup \Gamma$. There exist $\omega_1, \omega_2 \in \Omega$ such that $\|\eta_1 - \omega_1\| < \varepsilon$ and $\|\eta_2 - \omega_2\| < \varepsilon$. Let $\omega = \alpha\omega_1 + (1-\alpha)\omega_2$. Then $\omega \in \Omega$, and

$$
\begin{aligned}
d(\eta, \Omega) &\le \|\eta - \omega\| \\
&= \|\alpha(\eta_1 - \omega_1) + (1-\alpha)(\eta_2 - \omega_2)\| \\
&\le \alpha\|\eta_1 - \omega_1\| + (1-\alpha)\|\eta_2 - \omega_2\| \\
&< \varepsilon
\end{aligned}
$$

□

Theorem 4.2 For any integers n,k with $n \ge 2$ and $1 \le k \le n^2$, there exists a polytope $\mathcal{P}_k$ of dimension k containing a strictly unstable point and such that each min{k-1,m-1}-dimensional subpolytope is stable.

Proof Suppose a marginally unstable polytope $\bar{\mathcal{P}}_k$ of dimension k is constructed such that all its min{k-1,m-1} subpolytopes are stable. Then, since the set of stable points in $\mathbb{R}^{n\times n}$ is open and the union of all min{k-1,m-1}-dimensional subpolytopes of $\bar{\mathcal{P}}_k$ is compact, the subpolytopes of $\mathcal{P}_k = \bar{\mathcal{P}}_k + \varepsilon I$ of the same dimension are stable for sufficiently small ε, but $\mathcal{P}_k$ is strictly unstable. Thus, it suffices to construct any k-dimensional unstable $\mathcal{P}_k$ with stable subpolytopes.

If $n=2$, let $f(x,y)=\begin{bmatrix} 0 & x \\ -x & -1 \end{bmatrix}$, where x,y range over $\mathbb{R}$; otherwise, let

$$f(x,y)=\begin{bmatrix} 0 & 1 & -x^T \\ -1 & 0 & -y^T \\ x & y & -I \end{bmatrix}$$

where $x,y\in\mathbb{R}^{n-2}$. We consider two cases; first, assume $k<m$. Define $f_k:\mathbb{R}^k \longrightarrow \mathbb{R}^{n\times n}$ according to $f_k(\begin{bmatrix} w \\ 0 \end{bmatrix},\begin{bmatrix} z \\ 0 \end{bmatrix})$, where the vectors x and y are partitioned in any way such that $\begin{bmatrix} w \\ z \end{bmatrix}\in\mathbb{R}^k$. Since each f_k is affine and one-to-one, the set

$$\mathcal{Q}_k=\{f_k(w,z)\,\Big|\,\left\|\begin{bmatrix} w \\ z \end{bmatrix}\right\|_\infty\le 1\}$$

is a k-dimensional polytope. As in the proof of Theorem 3.1, each matrix in $\mathcal{Q}_k$ is stable except for the point corresponding to $w=z=0$. The union of the (k-1)-dimensional subpolytopes of $\mathcal{Q}_k$ is compact and nowhere dense in $f(\mathbb{R}^k)$; hence, there exist vectors $w_0,z_0\in f(\mathbb{R}^k)$ such that

$$\mathcal{P}_k=\{f_k(w+w_0,z+z_0)\,\Big|\,\left\|\begin{bmatrix} w \\ z \end{bmatrix}\right\|_\infty\le 1\}$$

is unstable, but has all its (k-1)-dimensional subpolytopes stable.

Next consider the case $k>m$. The union of the (m-1)-dimensional subpolytopes of the m-dimensional polytope $\mathcal{P}$ (defined in Theorem 3.1) are compact and nowhere dense in $f(\mathbb{R}^m)$; hence, there exist x_0,y_0 such that

$$\mathcal{O}=\{f(x-x_0,y-y_0)\,\Big|\,\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|_\infty\le 1\}$$

has all its (m-1)-dimensional subpolytopes stable, but $\mathcal{Q}$ is unstable. If n=2, let

$$g(x,y,z)=f(x,y)+\begin{bmatrix}z_1 & z_2\\ 0 & z_3\end{bmatrix}$$

Otherwise, define

$$g(x,y,z)=f(x,y)+\begin{bmatrix}z_1 & z_2 & z_3 & \cdots & z_n\\ z_{n+1} & z_{n+2} & z_{n+3} & \cdots & z_{2n}\\ 0 & 0 & z_{2n+1} & \cdots & z_{3n-2}\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & z_{n^2-3n+7} & \cdots & z_{n^2-2n+4}\end{bmatrix}$$

In each case, $z\in\mathbb{R}^{n^2-m}$. Also let $g_{k\varepsilon}(x,y,w)=g(x,y,\begin{bmatrix}(\varepsilon/2)w\\0\end{bmatrix})$, where $w\in\mathbb{R}^{k-2n+4}$ and $\varepsilon>0$. Note that each $g_{k\varepsilon}$ is affine and one-to-one.

Now consider the k-dimensional polytope

$$\mathcal{P}_{k\varepsilon}=\{g_{k\varepsilon}(x,y,w)\,\Big|\,\left\|\begin{bmatrix}x\\y\\w\end{bmatrix}\right\|_\infty\le 1\}$$

If we choose the matrix norm $\|M\|=\max|m_{ij}|$, it follows that for every vertex A of $\mathcal{P}_{k\varepsilon}$ there exists a vertex $\bar{A}$ of $\mathcal{Q}$ such that $\|A-\bar{A}\|<\varepsilon$. Furthermore, every (m-1)-dimensional subpolytope of $\mathcal{P}_{k\varepsilon}$ can be expressed as a disjoint finite union $\cup\Lambda_\nu$, where each Λ_ν is the convex hull of m-1 vertices $A_1,\ldots,A_{m-1}$ of $\mathcal{P}_{k\varepsilon}$. Suppose $A_1,\ldots,A_q\in\mathcal{Q}$ and $A_{q+1},\ldots,A_{m-1}\notin\mathcal{Q}$; let

$\Omega = \mathrm{conv}\{A_1, \ldots, A_q, \bar{A}_{q-1}, \ldots, \bar{A}_{m-1}\}$, where each $\bar{A}_i$ is a vertex of $\mathcal{C}$ satisfying $\|A_i - \bar{A}_i\| < \varepsilon$, and let $\Gamma = \{A_{q+1}, \ldots, A_{m-1}\}$. From Lemma 4.1, every $B \in \mathrm{conv}\{A_1, \ldots, A_{m-1}\} \subset \mathrm{conv}(\Omega \cup \Gamma)$ satisfies $d(B, \Omega) < \varepsilon$. Hence, for sufficiently small ε, each $(m-1)$-dimensional subpolytope of $\mathcal{P}_{k\varepsilon}$ is stable. $\square$

5. Conclusions

Our results demonstrate to what extent the techniques for checking polytope stability proposed in [2] can be extended to the case of n×n matrices. We have shown that, without further information describing the particular structure of a polytope, (2n-4)-dimensional faces must be checked for stability. Since testing even one such face can be a formidable task when n is large, and since the number of (2n-4)-dimensional faces grows exponentially with n, more work needs to be done before a computationally tractable algorithm can be devised for checking stability. It is our hope, however, that our work will be useful as an integral part of some future coherent theory of robust stability.

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THE MINIMAL DIMENSION OF STABLE FACES REQUIRED TO GUARANTEE STABILITY OF A MATRIX POLYTOPE: D-STABILITY[1]

J. Daniel Cobb

Department of Electrical and Computer Engineering
University of Wisconsin-Madison
1415 Johnson Drive
Madison, WI 53706-1691

Abstract

We consider the problem of determining whether a polytope $\mathcal{P}$ of $n\times n$ matrices is D-stable -- i.e. whether each point in $\mathcal{P}$ has all its eigenvalues in a given nonempty, open, convex, conjugate-symmetric subset D of the complex plane. Our approach is to check D-stability of certain faces of $\mathcal{P}$. In particular, for each D and n we determine the smallest integer m such that D-stability of every m-dimensional face guarantees D-stability of $\mathcal{P}$.

1. Introduction

Let $D\subset\mathbb{C}$ be nonempty, open, convex, and conjugate-symmetric (symmetric about the real axis), and define an $n\times n$ real matrix M to be D-stable if each eigenvalue λ of M satisfies $\lambda\in D$; otherwise, M is D-unstable. We consider the problem of determining whether certain subsets of $\mathbb{R}^{n\times n}$ consist entirely of D-stable matrices. To facilitate discussion we begin with some definitions.

A (convex) polytope $\mathcal{P}$ in a vector space V is the convex hull conv(Ω) of any nonempty finite subset $\Omega\subset V$. The dimension of $\mathcal{P}$ is the dimension of the affine hull aff($\mathcal{P}$) of $\mathcal{P}$. A face of $\mathcal{P}$ is any set of the form $\Pi\cap\mathcal{P}$, where Π is a supporting hyperplane of $\mathcal{P}$. Finally, a k-dimensional half-plane in V is any nonempty set of the form $\mathcal{H}=R\cap S$, where R is a closed half-space, S is a k-dimensional affine subspace, and $S\not\subset R$. (Note that this implies that aff($\mathcal{H}$) is simply S.)

In the robust control literature, considerable interest has been generated by the problem of determining whether a family of linear systems can be shown to consist entirely of D-stable systems by checking D-stability of certain representative members of that family. In many cases, such problems can be reduced to that of determining whether a polytope or other subset of $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$ consists entirely of D-stable points [1],[2]. (D-stability of a vector $x\in\mathbb{R}^n$ means simply that the polynomial $s^n+x_n s^{n-1}+\ldots+x_1$ has all its roots in D.) We are primarily interested in the technique of checking D-stability of lower dimensional faces of a polytope in order to guarantee D-stability of the entire set.

Most "facial" results pertain to continuous-time (CT) stability -- i.e. where D is the open left half complex plane. The seminal result [3] for polynomial polytopes motivates the approach. In [3] it is shown that a polynomial polytope of a particular simple structure (an "interval polynomial") is CT stable whenever four specially constructed vertices are CT stable. A more recent result [1] demonstrates that, for an arbitrary polynomial polytope, checking all edges is sufficient to guarantee CT stability. With respect to polytopes in $\mathbb{R}^{n\times n}$, is has been shown [4] that 1) an arbitrary polytope is CT stable if all (2n-4)-dimensional faces are CT stable and 2) there exist CT unstable polytopes such that all (2n-5)-dimensional faces are CT stable; hence, the value 2n-4 is minimal. In this paper we extend the results of [4] to D-stability where D may be any nonempty, open, convex conjugate-symmetric subset of $\mathbb{C}$.

We note that for the cases n=0 and n=1 our problem has a trivial solution: D-stability of vertices guarantees D-stability of the polytope. To handle $n\geq 2$ we need to partition the family of stability sets D according to the following two assumptions.

Assumption A. D is of the form $D=\{s\in\mathbb{C}\mid a<\operatorname{Re} s<b\}$, where $-\infty\leq a<b\leq\infty$

Assumption B: D is a nonempty, open, convex, conjugate-symmetric set not satisfying Assumption A.

In addition, we define $m_A(n)=\begin{cases}1, & n=2\\ 2n-4, & n>2\end{cases}$ and $m_B(n)=2n-2$. We intend to show that m_A and m_B are the values of m that we seek for cases A and B.

2. Sufficiency of m_A and m_B

Throughout our analysis, we will make extensive use of the fact that any affine, one-to-one map $f:\mathbb{R}^k\to\mathbb{R}^{n^2}$ determines an affine isomorphism between $\mathbb{R}^k$ and $f(\mathbb{R}^k)$. Among other things, this implies that, for any polytope $\mathcal{P}\subset\mathbb{R}^k$, $f(\mathcal{P})$ is also a polytope of the same dimension as $\mathcal{P}$; furthermore, f sets up a one-to-one correspondence between the q-dimensional faces of $\mathcal{P}$ and the q-dimensional faces of $f(\mathcal{P})$. In addition, f maps each k dimensional half-plane in $\mathbb{R}^k$ into another k-dimensional half-plane (e.g. see [5]). Finally, we note that every polytope is compact and that any set of the form $\{x\in\mathbb{R}^k \mid \|x\|_\infty\leq\gamma\}$, where $\gamma>0$, is a polytope whose q-dimensional faces are generated by fixing k-q entries of x at either $\pm\gamma$ and letting the remaining q entries vary independently over $[-\gamma,\gamma]$.

With these observations in mind, we prove a result characterizing the affine structure of the set of D-unstable points in $\mathbb{R}^{n\times n}$.

Lemma 2.1 1) If D satisfies Assumption A then for each D-unstable $M\in\mathbb{R}^{n\times n}$ there exists an (n^2-m_A)-dimensional half-plane $\mathcal{H}\subset\mathbb{R}^{n\times n}$ such that a) $M\in\mathcal{H}$ and b) $N\in\mathcal{H}$ implies N is D-unstable.

2) If D satisfies Assumption B, then for each D-unstable $M\in\mathbb{R}^{n\times n}$ there exists an (n^2-m_B)-dimensional half-plane $\mathcal{H}\subset\mathbb{R}^{n\times n}$ such that a) $M\in\mathcal{H}$ and b) $N\in\mathcal{H}$ implies N is D-unstable.

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Corollary: stability of every matrix on every $(2n-4)$-dimensional face of $\mathcal{P}$ guarantees stability of every matrix in $\mathcal{P}$.

Proof: If $\mathcal{P}$ contains an unstable A, there exists an (n^2-2n-4)-dimensional half-plane $\mathcal{H}$ consisting entirely of unstable points and containing A. From dimensionality arguments, such a plane must intersect a $(2n-4)$-dimensional face of $\mathcal{P}$. (See [4] for details.) □

3. Minimality of $m=2n-4$

In this section we show that, for every integer n, there exists a polytope $\mathcal{P}\subset\mathbb{R}^{n\times n}$ containing an unstable point and such that all $(2n-5)$-dimensional subpolytopes of $\mathcal{P}$ are stable. Hence, we conclude that checking stability of k-dimensional subpolytopes of $\mathcal{P}$, for any $k<2n-4$ is, in general, not sufficient to guarantee stability of $\mathcal{P}$.

Consider the polytope

$$\mathcal{P}=\left\{\begin{bmatrix} 0 & 1 & -x^T \\ -1 & 0 & -y^T \\ x & y & -I \end{bmatrix} \;\middle|\; \left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|_\infty \le 1\right\}$$

A routine calculation shows that $\mathcal{P}$ has characteristic polynomial $p(s)=(s+1)^{n-4}\Delta(s)$, where

$$\Delta(s)=s^4+2s^3+(2+x^Tx+y^Ty)s^2$$
$$+(2-x^Tx-y^Ty)s+1+x^Txy^Ty-(x^Ty)^2$$

From the Schwartz inequality, it is clear that all coefficients are strictly positive. The corresponding 4×4 Hurwitz matrix has its leading principal 3×3 minor equal to

$$M_3(x,y)=4x^Tx+4y^Ty+4(x^Ty)^2+(x^Tx-y^Ty)^2$$

Clearly, $M_3\ge0$ with equality iff $x=y=0$. Thus, $\mathcal{P}$ consists entirely of stable points, except for the relative interior point corresponding to $x=y=0$. We conclude that checking $(2n-5)$-dimensional faces (in this case the entire boundary of $\mathcal{P}$) is insufficient to guarantee stability.

Comments 1) The preceding example can be strengthened by adding ϵI to $\mathcal{P}$, where ϵ is sufficiently small. This yields a polytope with stable $(2n-5)$-dimensional boundary, but containing a ball of strictly unstable points.

2) Since the union of all $(2n-5)$-dimensional subpolytopes is nowhere dense, shifting the parameter set $\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|_\infty \le 1$ by an arbitrarily small vector yields an unstable polytope with all $(2n-5)$-dimensional subpolytopes stable.

3) The polytope $\mathcal{P}$ described above can be transformed into a similar example with any given dimension either by removing parameters or by using $\mathcal{P}$ as a face of a higher dimensional polytope.

Note that the constructions described in 1), 2), and 3) can be carried out simultaneously to given a stronger but algebraically messy version of the minimality proof offered above.

4. Conclusion

We have shown that $m=2n-4$ is the smallest integer such that stability of all m-dimensional subpolytopes of a given polytope $\mathcal{P}\subset\mathbb{R}^{n\times n}$ guarantees stability of $\mathcal{P}$. Furthermore, we have demonstrated that checking m dimensional faces is always sufficient. This reduces the task of determining whether a polytope is stable to that of deciding whether several low-dimensional polytopes are stable. Our result has certain theoretical significance; however, more work needs to be done before it can be decided whether the result will help to reduce the computational burden inherent in robust system design.

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With these observations in mind, we prove a result characterizing the affine structure of the set of D-unstable points in $\mathbb{R}^{n\times n}$.

<u>Lemma 2.1</u> If D satisfies Assumption A (respectively, Assumption B), then for each D-unstable $M\in\mathbb{R}^{n\times n}$ there exists an (n^2-m_A)-dimensional (respectively, (n^2-m_B)-dimensional) half-plane $\mathcal{H}\subset\mathbb{R}^{n\times n}$ such that a) $M\in\mathcal{H}$ and b) $N\in\mathcal{H}$ implies N is D-unstable.

<u>Proof</u> Suppose Assumption A holds. If $a=-\infty, b=\infty$, the statement is vacuously true; otherwise, we need to consider two cases.

Case I -- M has a real eigenvalue $\lambda_0\notin D$: Let $T=[v\ W]$, where v is an eigenvector corresponding to λ_0 and W is chosen to make T nonsingular. Clearly, the map $f:\mathbb{R}^{n^2-n+1}\longrightarrow\mathbb{R}^{n\times n}$ determined by $f(\lambda,y,Z)=T\begin{bmatrix}\lambda & y\\ 0 & Z\end{bmatrix}T^{-1}$ is affine and one-to-one. If $\lambda_0<a$, let $I=(-\infty,\lambda_0]$ and let $\bar{\mathcal{H}}$ be the (n^2-n+1)-dimensional half-plane $\bar{\mathcal{H}}=\{f(\lambda,y,Z)\mid \lambda\in I, y\in\mathbb{R}^{1\times n-1}, Z\in\mathbb{R}^{n-1\times n-1}\}$. If $a=-\infty$, then $\lambda_0>b$ so we set $I=[\lambda_0,\infty)$ and construct $\bar{\mathcal{H}}$ in the same way. In either case, $M\in\bar{\mathcal{H}}$ and every matrix in $\bar{\mathcal{H}}$ is D-unstable. Since $n^2-n+1\ge n^2-m_A$, it remains to select any (n^2-m_A)-dimensional half-plane $\mathcal{H}$ satisfying $M\in\mathcal{H}\subset\bar{\mathcal{H}}$.

Case II -- M has a complex eigenvalue pair $\alpha_0\pm i\beta_0$ with $\alpha_0>0$: Let $T=[u\ v\ W]$, where u+iv is an eigenvector corresponding to $\alpha_0+i\beta_0$ and W is chosen to make T nonsingular. If $\alpha_0<a$, let $\bar{\mathcal{H}}$ be the (n^2-2n+4)-dimensional

half plane $\bar{\mathcal{H}}=\{T\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}T^{-1} \mid \mathrm{tr}\, X \leq 2\alpha_0, Y\in\mathbb{R}^{2\times n-2}, Z\in\mathbb{R}^{n-2\times n-2}\}$. ($\mathrm{tr}\, X \leq 2\alpha_0$ describes a 4-dimensional half-plane, since $\mathrm{tr}\, X = \langle X, I \rangle$.) $\bar{\mathcal{H}}$ contains only D-unstable points, since $\mathrm{tr}\, X \leq 2\alpha_0$ implies X has at least one eigenvalue λ with $\mathrm{Re}\,\lambda \leq \alpha_0$. If $\alpha_0 > b$, $\bar{\mathcal{H}}$ is defined by $\mathrm{tr}\, X \geq 2\alpha_0$, and the same reasoning holds. In either case, $M\in\bar{\mathcal{H}}$.

Now suppose Assumption B holds. We again consider two cases. Case I -- M has a real eigenvalue $\lambda_0 \notin D$: Again let $T=[v\ W]$, where v is an eigenvector corresponding to λ_0. Since D is convex, either $(-\infty, \lambda_0] \cap D = \phi$ or $[\lambda_0, \infty) \cap D = \phi$. In the former case, let $\bar{\mathcal{H}}$ be the (n^2-n+1)-dimensional half-plane $\bar{\mathcal{H}}=\{T\begin{bmatrix} \lambda & y \\ 0 & Z \end{bmatrix}T^{-1} \mid \lambda \leq \lambda_0, y\in\mathbb{R}^{1\times n-1}, Z\in\mathbb{R}^{n-1\times n-1}\}$. For the latter case, alter the definition of $\bar{\mathcal{H}}$ by substituting "$\lambda \geq \lambda_0$" for "$\lambda \leq \lambda_0$". In either case, $M\in\bar{\mathcal{H}}$ and every matrix in $\bar{\mathcal{H}}$ is D-unstable. Since $n^2-n+1 > n^2-m_B$, it remains to select any (n^2-m_B)-dimensional half-plane $\mathcal{H}$ satisfying $M\in\mathcal{H}\subset\bar{\mathcal{H}}$.

Case II -- M has a complex eigenvalue pair $\alpha_0 \pm i\beta_0 \notin D$: Let $T=[u\ v\ W]$, where $u+iv$ is an eigenvector corresponding to $\alpha_0+i\beta_0$. Since D is convex, there exists a half-space $\Pi\subset\mathbb{C}$ such that $\alpha_0+i\beta_0\in\Pi$ and $\Pi\cap D=\phi$. Let $\mathcal{H}$ be the (n^2-2n+2)-dimensional half-plane

$$\mathcal{H}=\{T\begin{bmatrix} \alpha & \beta & x \\ -\beta & \alpha & y \\ 0 & 0 & Z \end{bmatrix}T^{-1} \mid \alpha+i\beta\in\Pi; x,y\in\mathbb{R}^{1\times n-2}; Z\in\mathbb{R}^{n-2\times n-2}\}$$

Clearly, $\mathcal{H}$ contains only D-unstable points, and $M\in\mathcal{H}$. □

Next we prove an easy result concerning the intersection of affine sets.

<u>Lemma 2.2</u> Let V be a p-dimensional Euclidean space, $\mathcal{H}\subset V$ a k-dimensional half-plane, and Γ a q-dimensional affine subspace with $k+q>p$. Consider any vector $x_0\in\mathcal{H}\cap\Gamma$. There exists a $(k+q-p)$-dimensional half-plane $\bar{\mathcal{H}}$ such that $x_0\in\bar{\mathcal{H}}\subset\mathcal{H}\cap\Gamma$.

Proof By definition, $\mathcal{H}=R\cap S$, where R is a closed half-space and S is a k-dimensional affine subspace satisfying $S\not\subset R$. There exists an affine subspace $\bar{S}\subset S\cap\Gamma$ with $\dim\bar{S}=k+q-p$ and $x_0\in\bar{S}$. If $\bar{S}\subset R$, let $\bar{\mathcal{H}}\subset R\cap\bar{S}$ be any $(k+q-p)$-dimensional half-space containing x_0. Then $\bar{\mathcal{H}}\subset R\cap S\cap\Gamma=\mathcal{H}\cap\Gamma$. If $\bar{S}\not\subset R$, let $\bar{\mathcal{H}}=R\cap\bar{S}$. Then $x_0\in\bar{\mathcal{H}}$, since $x_0\in\mathcal{H}\cap\Gamma\subset R$. Also, $\dim\bar{\mathcal{H}}=k+q-p$, since $\bar{\mathcal{H}}$ is nonempty. □

We are now in a position to prove our first main result.

Theorem 2.3 Under Assumption A (respectively, Assumption B), D-stability of every matrix in every m_A-dimensional (respectively, m_B-dimensional) face of $\mathcal{P}$ guarantees D-stability of every matrix in $\mathcal{P}$.

Proof Suppose Assumption A holds. Our arguments here are similar to those used in [2, Lemma 1]. If $\mathcal{P}_k$ is a D-unstable polytope of dimension $k>m_A$, there exists a D-unstable matrix $M_1\in\mathcal{P}_k$. From Lemma 2.1, there is an (n^2-m_A)-dimensional half plane $\mathcal{H}_1$, consisting entirely of D-unstable points and containing M_1. Since $\mathcal{H}_1$ is unbounded, there exists an $M_2\in\mathcal{H}_1$ lying on the boundary of $\mathcal{P}_k$ and, hence, in one of its $(k-1)$-dimensional faces $\mathcal{P}_{k-1}$. From Lemma 2.2, the intersection $\mathcal{H}_1\cap\mathrm{aff}(\mathcal{P}_{k-1})$ contains a $(k-m_A-1)$-dimensional half-plane $\mathcal{H}_2$ such that $M_2\in\mathcal{H}_2$. Proceeding inductively, we find that there exists an m_A-dimensional face $\mathcal{P}_m$ and a point $M_{k-m}\in\mathcal{P}_m$ such that M_{k-m} is D-unstable.

Under Assumption B, the same proof holds if we replace m_A by m_B. □

3. Minimality of m_A and m_B

Our next task is to show that m_A and m_B are the smallest integers such that D-stability of all m_A-dimensional or m_B-dimensional faces of $\mathcal{P}$ guarantees D-stability of $\mathcal{P}$ under Assumptions A or B, respectively. In order to prove

this we need a lemma which may be interpreted as a multivariable extension of L'Hospital's rule. For any k×k matrices Q and R, we use the notation $Q>0$ and $R<0$ to signify that Q is positive definite and R is negative definite, respectively.

<u>Lemma 3.1</u> Let $0\in U\subset \mathbb{R}^k$ with U open, and let $e_1, e_2: U \longrightarrow \mathbb{R}^2$ be C^2 functions. In addition, suppose $e_1(0)=e_2(0)=0$,

$$\left.\frac{\partial e_1}{\partial x}\right|_{x=0} = \left.\frac{\partial e_2}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial^2 e_1}{\partial x^2}\right|_{x=0} = 0, \quad \left.\frac{\partial^2 e_2}{\partial x^2}\right|_{x=0} < 0$$

For every $\delta>0$ there exists an $\varepsilon>0$ such that $0\neq\|x\|<\varepsilon$ implies $e_2(x)<-\frac{1}{\delta}|e_1(x)|$.

<u>Proof</u> From [6,p.340], for every $Q>0$ there exists an $\varepsilon>0$ such that $\|x\|<\varepsilon$ implies

$$\frac{\left|e_1(x) - e_1(0) - \left.\frac{\partial e_1}{\partial x}\right|_{x=0} x - \frac{1}{2}x^T\left.\frac{\partial^2 e_1}{\partial x^2}\right|_{x=0} x\right|}{x^TQx} < \frac{1}{2}\left(\frac{\delta}{1+\delta}\right)$$

Setting $Q = -\left.\frac{\partial^2 e_2}{\partial x^2}\right|_{x=0}$ yields

$$\frac{|e_1(x)|}{x^TQx} < \delta, \quad \frac{\left|e_2(x)+\frac{1}{2}x^TQx\right|}{x^TQx} < \delta \tag{1}$$

and, from (1), $e_2(x) < (\delta-\frac{1}{2})x^TQx < 0$ for $x\neq0$. Hence, for $x\neq0$,

$$\left|\frac{e_1(x)}{e_2(x)}\right| = \frac{|e_1(x)|}{\left|(e_2(x)+\frac{1}{2}x^TQx) - \frac{1}{2}x^TQx\right|} = \frac{\frac{|e_1(x)|}{x^TQx}}{\left|\frac{e_2(x)+\frac{1}{2}x^TQx}{x^TQx} - \frac{1}{2}\right|} < \frac{\frac{1}{2}\left(\frac{\delta}{1+\delta}\right)}{\frac{1}{2} - \frac{1}{2}\left(\frac{\delta}{1+\delta}\right)} = \delta$$

Thus, $e_2(x)<-\frac{1}{\delta}|e_1(x)|$. $\square$

Now we can prove our second main result.

Theorem 3.2 Suppose D satisfies Assumption A (respectively, Assumption B). For each n there exists an m_A dimensional (respectively, m_B-dimensional) polytope $\mathcal{P}\subset\mathbb{R}^{n\times n}$ containing a D-unstable point and such that all (m_A-1)-dimensional (respectively, (m_B-1)-dimensional) faces of $\mathcal{P}$ are D-stable.

Proof Under Assumption A we need to consider nine cases.

Case I -- $a>-\infty, b<\infty, n=2$: Consider the affine, one-to-one map

$$f(x) = \begin{bmatrix} b & x \\ -x & \frac{a+b}{2} \end{bmatrix}$$

and the corresponding one-dimensional polytope $\mathcal{P}=\{f(x) \,\big|\, |x|\le 1\}$. The point in $\mathcal{P}$ corresponding to $x=0$ is clearly D-unstable. It suffices to prove that the characteristic polynomials Δ^+ and Δ^- of $f(x)-bI$ and $aI-f(x)$, respectively, are Hurwitz for all $x\neq 0$. This is in fact true, since $\Delta^+(s)=s^2+\frac{b-a}{2}s+x^2$ and $\Delta^-(s)=s^2+\frac{3}{2}(b-a)s+\frac{(b-a)^2}{2}+x^2$ have positive coefficients for $x\neq 0$.

Case II -- $a>-\infty, b<\infty, n=3$: Let

$$f(x,y) = \begin{bmatrix} b & 1 & -x \\ -1 & b & -y \\ x & y & \frac{a+b}{2} \end{bmatrix}$$

It is sufficient to show that the characteristic polynomials Δ^+ and Δ^- of $f(x,y)-bI$ and $aI-f(x,y)$, respectively, are Hurwitz for $\begin{bmatrix} x \\ y \end{bmatrix}\neq 0$. A straightforward calculation yields $\Delta^+(s)=s^3+\frac{b-a}{2}s^2+(1+x^2+y^2)s+\frac{b-a}{2}$ and $\Delta^-(s)=s^3+\frac{5}{2}(b-a)s^2+(1+x^2+y^2+2(b-a)^2)s+\frac{b-a}{2}(1+2x^2+2y^2+(b-a)^2)$. Each polynomial has positive coefficients for $\begin{bmatrix} x \\ y \end{bmatrix}\neq 0$. The fact that they are Hurwitz follows from positivity of the second-order leading principal minors $M_2^+ = \frac{b-a}{2}(x^2+y^2)$ and $M_2^- = \frac{b-a}{2}(4+3x^2+3y^2+9(b-a)^2)$ of the corresponding 3×3 Hurwitz matrices.

Case III -- $b>-\infty, a<\infty, n>3$: Let $f(x,y) = \begin{bmatrix} b & 1 & -x^T \\ -1 & b & -y^T \\ x & y & (\frac{a+b}{2})I \end{bmatrix}$

where $x,y,\in\mathbb{R}^{n-2}$. A tedious calculation shows that $\Delta^+(s) = (s+\frac{b-a}{2})^{n-4}\hat{\Delta}^+(s)$ and

$$\Delta(s) = (s+\frac{b-a}{2})^{n-4}\hat{\Delta}(s), \text{ where}$$

$$\hat{\Delta}^{+}(s) = s^4+(b-a)s^3+(1+x^Tx+y^Ty+(\frac{b-a}{2})^2)s^2+\frac{b-a}{2}(2+x^Tx+y^Ty)s+x^Txy^Ty-(x^Ty)^2+(\frac{3-\alpha}{\alpha})^2$$

$$\hat{\Delta}^{-}(s) = s^4+3(b-a)s^3+(1+x^Tx+y^Ty+\frac{13}{4}(b-a)^2)s^2+(b-a)(1+\frac{3}{2}x^Tx+\frac{3}{2}y^Ty+\frac{3}{2}(b-a)^2)s$$
$$+x^Txy^Ty-(x^Ty)^2+(\frac{b-a}{2})^2(1+2x^Tx+2y^Ty+(b-a)^2)$$

From the Schwartz inequality, $\hat{\Delta}^{+}$ and $\hat{\Delta}^{-}$ have positive coefficients when $\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$. Furthermore, the third-order leading principal minors of the corresponding 4×4 Hurwitz matrices of $\hat{\Delta}^{+}$ and $\hat{\Delta}^{-}$ are

$$M_3^{+}(s) = (\frac{b-a}{2})^2(2(1+(\frac{b-a}{2})^2)(x^Tx+y^Ty)+(x^Tx-y^Ty)^2+4(x^Ty)^2)$$

$$M_3^{-}(s) = (2+\frac{9}{2}x^Tx+\frac{9}{2}y^Ty+\frac{9}{4}(x^Tx-y^Ty)^2+(x^Ty)^2)(b-a)^2$$
$$+(\frac{27}{4}+\frac{63}{8}x^Tx+\frac{63}{8}y^Ty)(b-a)^4+\frac{81}{8}(b-a)^6$$

Since M_3^{+} and M_3^{-} are positive, $\hat{\Delta}^{+}$ and $\hat{\Delta}^{-}$ and, hence, Δ^{+} and Δ^{-} are Hurwitz.

The remaining six cases are handled similarly by choosing all eigenvalues in the interior of D, except for one or two on the boundary of D. For example, for $a>-\infty$, $b=\infty$, $n>3$, set

$$f(x,y) = \begin{bmatrix} a & 1 & -x^T \\ -1 & a & -y^T \\ x & y & (a+1)I \end{bmatrix}$$

Adopting Assumption B, suppose D is not of the form $\{s \mid a<\text{Re } s<b\}$. Since D is convex, there exists a real $\alpha_0 \in D$ such that the line $L=\{\alpha_0+i\beta \mid \beta \in \mathbb{R}\}$ satisfies $L \not\subset D$. Since D is conjugate symmetric and open, there exists a $\beta_0>0$ such that $\alpha_0 \pm i\beta_0$ are boundary points of D, but $\alpha_0 \pm i\beta \in D$ when $|\beta|<\beta_0$. Furthermore, there

exists a $\delta>0$ such that $\alpha_0\pm\delta\in D$. Again invoking convexity, the open diamond $\bar{d}_\delta$=int conv$\{\alpha_0\pm\delta\beta_0,\alpha_0\pm i\beta_0\}$ is contained in D. To simplify the problem, consider the open diamond $d_\delta=\frac{1}{\beta_0}(\bar{d}_\delta-\alpha_0)$=int conv$\{\pm i,\pm\delta\}$. We need only construct a single polytope $\mathcal{P}$ containing a matrix with a pair of eigenvalues at $\pm i$ and with all m_B-dimensional faces consisting of matrices with all eigenvalues in d_δ: then $\beta_0\mathcal{P}+\alpha_0 I$ would then satisfy the desired properties with respect to $\bar{d}_\delta$.

Consider the (n^2-2n+2)-dimensional polytope

$$\mathcal{P}_\varepsilon = \left\{\begin{bmatrix} w & 1+x & y^T \\ -1+x & -w & z^T \\ y & z & 0 \end{bmatrix} \middle| \left\|\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}\right\|_\infty \le \varepsilon\right\}$$

where $y,z\in\mathbb{R}^{n-2}$. Clearly, $\mathcal{P}_\varepsilon$ has a D-unstable point M at w=x=0, y=z=0. We will show that for sufficiently small ε, every pint in $\mathcal{P}_\varepsilon$ except M is D-stable. Hence, $\mathcal{P}=\mathcal{P}_\varepsilon$ satisfies the desired properties.

Case I -- n=2: Each point in $\mathcal{P}_\varepsilon$ has characteristic polynomial $\Delta(w,x,s)=s^2+1-w^2-x^2$ and hence has eigenvalues $\pm i(1-w^2-x^2)^{1/2}$. Let $\varepsilon<(\frac{(1+\delta)^2}{2})^{1/2}$.

Case II -- n=3: Each point in $\mathcal{P}_\varepsilon$ has characteristic polynomial

$$\Delta(w,x,y,z,s) = s^3+(1-w^2-x^2-y^2-z^2)s-(w(y^2-z^2)+2xyz)$$

Let $g(w,x,y,z,\alpha,\beta) = \begin{bmatrix}\mathrm{Re}\ \Delta(w,x,y,z,\alpha+i\beta) \\ \mathrm{Im}\ \Delta(w,x,y,z,\alpha+i\beta)\end{bmatrix}$. It is easy to see that g is a polynomial function and, hence, analytic. A straightforward calculation shows

$$\left.\frac{\partial g}{\partial(\alpha,\beta)}\right|_{\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}} = \begin{bmatrix}-2 & 0 \\ 0 & -1\end{bmatrix}$$

Thus, from the implicit function theorem, there exists a unique analytic function $h:U \longrightarrow \mathbb{R}^2$ such that $h(0)=\begin{bmatrix}0\\1\end{bmatrix}$ and $g(w,x,y,z,h(w,x,y,z))=0$ for every $[w\ x\ y\ z]^T \in U$.

Next, let $\begin{bmatrix}e_1\\e_2\end{bmatrix}=h-\begin{bmatrix}0\\1\end{bmatrix}$. A tedious computation shows

$$\left.\frac{\partial e_1}{\partial(w,x,y,z)}\right|_{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}} = \left.\frac{\partial e_2}{\partial(w,x,y,z)}\right|_{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}} = 0, \quad \left.\frac{\partial^2 e_1}{\partial(w,x,y,z)^2}\right|_{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}} = 0, \quad \left.\frac{\partial^2 e_2}{\partial(w,x,y,z)^2}\right|_{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}} = -I$$

From Lemma 3.1, there exists an $\epsilon>0$ such that $e_2(w,x,y,z)<-\frac{1}{\delta}|e_1(w,x,y,z)$ whenever $0\neq\|[w\ x\ y\ z]^T\|<\epsilon$. Since e_1 and e_2 are continuous, we may also assume $|e_i|<1;\ i=1,2.$ Returning to $h=\begin{bmatrix}h_1\\h_2\end{bmatrix}$, it follows that $h_2(w,x,y,z)<1-\frac{1}{\delta}|h_1(w,x,y,z)|$, $|h_2(w,x,y,z)|<1$, and $|h_1(w,x,y,z)-1|<1$ for all $[w\ x\ y\ z]^T\neq 0$. Hence, $h\in d_\delta$.

Case III -- n>3: We have

$$\Delta(w,x,y,z,s) = s^4+(1-w^2-x^2-x^Tx-y^Ty)s^2-(wy^Ty-2xy^Tz-wz^Tz)s+y^Tyz^Tz-(y^Tz)^2$$

Let $g(w,x,y,z,\alpha,\beta) = \begin{bmatrix}\mathrm{Re}\ \Delta(w,x,y,z,\alpha+i\beta)\\ \mathrm{Im}\ \Delta(w,x,y,z,\alpha+i\beta)\end{bmatrix}$. Again, g is a polynomial function; in this case

$$\left.\frac{\partial g}{\partial(\alpha,\beta)}\right|_{\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\end{bmatrix}} = \begin{bmatrix}0 & -2\\2 & 0\end{bmatrix}$$

Thus there exists an open $U\subset\mathbb{R}^{2n-2}$ with $0\in U$ and $h:U \longrightarrow \mathbb{R}^2$ such that $h(0)=\begin{bmatrix}0\\1\end{bmatrix}$ and $g(w,x,y,z,h(w,x,y,z))=0$ for every $[w\ x\ y\ z]^T\in U$. Let $\begin{bmatrix}e_1\\e_2\end{bmatrix}=h-\begin{bmatrix}0\\1\end{bmatrix}$. Then

$$\left.\frac{\partial e_1}{\partial(w,x,y,z)}\right|_{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}} = \left.\frac{\partial e_2}{\partial(w,x,y,z)}\right|_{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}} = 0, \quad \left.\frac{\partial^2 e_1}{\partial(w,x,y,z)^2}\right|_{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}} = 0, \quad \left.\frac{\partial^2 e_2}{\partial(w,x,y,z)^2}\right|_{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}} = -I$$

Applying Lemma 3.1 as in Case II, it follows that $h(w,x,y,z) \in d_\delta$ for every $[w\ x\ y\ z]^T \neq 0$.

Note that Theorem 3.2 also implies that the half-planes constructed in Lemma 2.1 are maximal in the sense that there exists a D-unstable matrix M in $\mathbb{R}^{n\times n}$ such that every half-plane containing M of dimension greater than n^2-m_A or n^2-m_B must also contain a D-unstable matrix. Indeed, if this were not the case, the arguments in Theorem 3.2 could be used to prove that m_A and m_B are not minimal.

4. Conclusions

Our results demonstrate to what extent the techniques for checking polytope stability proposed in [1] and [3] can be extended to the case of n×n matrices. We have shown that, without further information describing the particular structure of a polytope, either (2n-4)-dimensional or (2n-2)-dimensional faces need to be checked for D-stability, depending on the structure of D. Since testing even one such face can be a formidable task when n is large, and since the number of (2n-4)-dimensional and (2n-2)-dimensional faces grow exponentially with n, more work needs to be done before a computationally tractable algorithm can be devised for checking D-stability. It is our hope, however, that our work will be useful as an integral part of some future coherent theory of robust stability.

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A REALIZATION THEORY FOR PERTURBED LINEAR SYSTEMS[1]

Mingde Tan
J. Daniel Cobb

Department of Electrical and Computer Engineering
University of Wisconsin
1415 Johnson Drive
Madison, WI 53706-1691

Abstract

In this paper we present a theory which characterizes LTI state-space realizations of perturbed rational transfer function matrices. Our approach is to model system perturbations as sequences in the space of rational matrices. First, we give a definition of convergence in the space of rational matrices which is motivated by the kinds of parameter uncertainties occurring in many robust control problems. A realization theory is then established under the constraint that the realization of any convergent sequence of rational matrices should also be convergent. Next, we consider the issue of minimality of realizations and propose a method for calculating the dimension of a minimal realization of a given transfer matrix sequence. Finally, necessary and sufficient conditions are discussed under which a sequence of state-space systems is a minimal realization and under which minimal realizations of the same transfer function sequence are state-space equivalent. Relationships with standard algebraic system theoretic results are discussed.

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1. Introduction

The theory of state-space realizations for strictly proper rational matrices has been thoroughly studied (e.g. see [16]). More recently, techniques for handling improper transfer matrices have been devised (see [14]). In this paper we extend those ideas to the perturbational case -- i.e. where a system is described by a convergent sequence of rational matrices (possibly improper). A realization is then a sequence of (generalized) state-space systems. The problem is made nontrivial by imposing the constraint that the matrix entries of the realization sequence should also converge.

Part of our motivation for this problem comes from the study of robust control problems -- specifically from those dealing with order uncertainty and singular perturbations. For example, the robustness problems addressed in [1]-[3] are based on singularly perturbed system models. Physical systems are invariably subject to some variations in parameters, often resulting in changes in model order. It is desirable, therefore, to design compensators which meet performance criteria independent of system perturbations. Many robust control theories (e.g. [4]) emphasize input-output performance characteristics. Our intention is to develop some fundamental tools for examining robustness problems associated with a system's internal structural properties.

One way to approach this problem might be through the application of algebraic system theory (see,e.g.,[17]). In this setting, the transfer function sequence is viewed as a rational function over the ring c of convergent real sequences using pointwise operations. Unfortunately, we will see that existing results in algebraic realization theory apply to our case only marginally. This is due to three key facts: 1) The ring c is not an

integral domain 2) Most results in algebraic realization theory deal only with the case of proper transfer functions. 3) An abstract version of the Weierstrass decomposition for matrix pencils over a ring does not yet exist. Nevertheless, our feeling is that the properties of sequences of transfer functions are sufficiently important from the point of view of robust control theory that they deserve separate treatment, not only for the sake of mimicking standard results from the algebraic theory, but also in order to obtain deeper insight into the specific structure of realizations over this particular ring.

From an analytic perspective, considerable work dealing with perturbations of rational matrices has appeared (e.g., [4],[7]-[12]). In these papers various rational matrix topologies have been proposed, motivated by a variety of control problems. The closest of these to our work are [10]-[12], where a singular perturbation theory for transfer functions is developed and a specific form of realization is given. However, [10]-[12] do not explicitly address those problems dealing with the existence of realizations in general and, in particular, the minimal realization of perturbed systems. In [4] rational matrix convergence is characterized in terms of the "graph metric" which is used to address certain problems in local simultaneous stabilization. It is easy to show that the graph metric induces a topology which is very different from that corresponding to simple system parameter convergence. The work of [7] and [9] also treats the problem of topologizing the set of rational matrices and is closely related to ours, but again does not examine the realization problem. Our work is motivated solely by realization and robustness issues; our constructions are designed to yield the simplest definition of convergence corresponding to convergence of system parameters.

We are mainly concerned with the interplay between two types of LTI system representations. First, let $\mathbb{R}(s)$ be the set of all rational functions over $\mathbb{R}$, and let $\mathbb{R}(s)^{rm}$ be the set of $r\times m$ matrices over $\mathbb{R}(s)$. Next, consider (generalized) state-space systems

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{1}$$

where E and A are $n\times n$ real matrices satisfying the standard regularity assumption $\det(sE-A)\neq 0$, B is $n\times m$, C is $r\times n$, and E may be singular. For the sake of brevity, we identify the system (1) with the matrix 4-tuple $\sigma=(E,A,B,C)\in\mathbb{R}^{n(2n+m+r)}$. The transfer matrix of (1) is

$$H(s) = C(sE-A)^{-1}B = \frac{C\cdot\mathrm{adj}(sE-A)\cdot B}{\det(sE-A)} \in \mathbb{R}(s)^{rm}. \tag{2}$$

Throughout the paper we assume that the values of m and r are fixed; we consider n to be a variable.

Definition 1.1 1) A state-space system $\sigma\in\mathbb{R}^{n(n+m+r)}$ is said to have dimension n. In this case, we write $\dim\sigma=n$.

2) If a rational matrix H is of the form (2), we say that (E,A,B,C) is a realization of H.

With regard to (1) and (2), a (nonperturbational) realization theory already appears in [14]. We now summarize the main results of this theory.

Theorem 1.2 [14]

1) Every rational matrix has a realization.

2) The minimal dimension over all realizations of H, denoted $\mu(\cdot)$, is

$$\mu(H(s)) = \nu(H_s(s)) + \nu(\tfrac{1}{s}H_f(\tfrac{1}{s})),$$

where $\nu(\cdot)$ is MacMillan degree, and H_s and H_f are the unique strictly proper rational matrix and polynomial matrix, respectively, satisfying $H=H_s+H_f$.

3) A 4-tuple σ is a minimal realization of some rational matrix H if and only if σ is controllable and observable (as defined in [8]).

4) If $\sigma_1=(E_1,A_1,B_1,C_1)$ and $\sigma_2=(E_2,A_2,B_2,C_2)$ are minimal realizations of the same rational matrix, there exist nonsingular matrices M and N such that $E_2=ME_1N$, $A_2=MA_1N$, $B_2=MB_1$, and $C_2=C_1N$.

The results of our present work may be considered to be a generalization of Theorem 1.2 to the case of rational matrix sequences $\{H_k\}$.

In Section 2 we will choose a natural definition for convergence of rational matrices. Working from this definition, we will consider sequences $H_k \rightarrow H$ in $\mathbb{R}(s)^{rm}$ and attempt to characterize those sequences $\{\sigma_k\}$ in $\mathbb{R}^{n(2n+m+r)}$ such that 1) σ_k converges to some σ in the matrix sense, 2) σ_k is a realization of H_k for sufficiently large k, and 3) σ is a realization of H. We view this approach as a way of modeling the possible perturbations in the internal structure of a system corresponding to a given perturbation in the input-output description $\{H_k\}$.

In our realization theory, we will see that Theorem 1.2, part 1) remains true (Section 5). Corresponding to the expression for μ in part 2), in Section 6 we will define and give an explicit expression for a degree function which equals the dimension of all "minimal realizations" of a sequence of transfer matrices. It will be shown that properties 3) and 4) do not hold as stated for sequences; however, we will discuss important special cases where similar statements do hold. In Sections 5 and 6 we will also discuss the connections between our work and the standard algebraic realization theory (see [17]).

2. Convergence in the Space of Rational Matrices

We first consider the problem of defining a topology on $\mathbb{R}(s)$ and later on $\mathbb{R}(s)^{rm}$. Convergence in $\mathbb{R}(s)$ will be defined in the most natural way such that small perturbations in $\mathbb{R}(s)$ correspond to small perturbations in the coefficients of numerator and denominator of the rational function. To begin, we must define convergence in the set $\mathbb{R}[s]$ of all polynomials over $\mathbb{R}$. Suppose $P_k: k=1,2,\ldots$ and P are polynomials in $\mathbb{R}[s]$.

<u>Definition 2.1</u> We say <u>P_k converges to P</u> if there exists an integer $q<\infty$ such that $\deg P_k \le q$; $k=1,2,\ldots$, $\deg P \le q$, and $a_{ik} \to a_i$: $i=0,\ldots,q$, where

$$P_k(s) = a_{qk}s^q + \cdots + a_{1k}s + a_{0k}; \; k=1,2,\ldots$$
$$P(s) = a_q s^q + \cdots + a_1 s + a_0.$$

<u>Remarks</u>

1) If we regard $P_k \in \mathbb{R}[s]$; $k=1,2,\cdots$ as functions over $\mathbb{C}$, we might be tempted to define $P_k \to P$ when $\lim_{k\to\infty} P_k(s)=P(s)$ for any $s\in\mathbb{C}$. But we notice that in this definition, $\deg P_k$ may not be bounded. For example, let $P_k(s)=\frac{1}{k^k}s^k+1$. This observation brings us to a crossroads in the theory: If we were to allow convergent polynomial sequences to have unbounded degree, the same would be true for sequences of rational functions. This would result in an undesirable situation where state-space realizations could have unbounded dimension. Hence, we insist on bounded degree based both on physical intuition and on a desire for mathematical elegance.

2) Definition 2.1 is equivalent to the following two conditions:

   a) $\{\deg P_k \mid k=1,2,\cdots\}$ is bounded.

   b) $\lim_{k\to\infty} P_k(s)=P(s)$ for every $s \in \mathbb{C}$.

   Indeed, necessity of a) and b) is obvious. On the other hand, if $\{P_k\}$ satisfies a), $P_k(s)$ and $P(s)$ can be written as in Definition 2.1. Choose

$q-1$ distinct complex numbers $\{s_1, \ldots s_{q-1}\}$. Then, from b),

$$V\xi_k \longrightarrow V\xi$$

where

$$V = \begin{bmatrix} s_1^q & \cdots & s_1 & 1 \\ s_2^q & \cdots & s_2 & 1 \\ \vdots & & \vdots & \vdots \\ s_{q-1}^q & \cdots & s_{q-1} & 1 \end{bmatrix}, \quad \xi_k = \begin{bmatrix} a_{qk} \\ \vdots \\ a_{1k} \\ a_{0k} \end{bmatrix}, \quad \xi = \begin{bmatrix} a_q \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}.$$

We know that the Vandermonde matrix V satisfies det $V \neq 0$ as long as $s_i \neq s_j$, $i \neq j$: therefore V^{-1} exists, and $\xi_k \longrightarrow \xi$ as $k \longrightarrow \infty$.

3) We can define a topology on $\mathbb{R}[s]$ which is consistent with our notion of convergence in Definition 2.1. To do so, identify every element in $\mathbb{R}[s]$ with an element in $\mathbb{R}^\infty$ according to

$$p_m s^m + \cdots + p_1 s + p_0 \longleftrightarrow (p_0, p_1, \cdots, p_m, 0, 0, 0, \cdots),$$

and let

$$\mathcal{R}_{m+1} = \{(p_0, p_1, \cdots, p_m, 0, 0, \cdots) \in \mathbb{R}^\infty \mid p_i \in \mathbb{R};\ i = 0, 1, 2, \cdots, m\}.$$

Then

$$\mathcal{R} = \bigcup_{k=1}^{\infty} \mathcal{R}_k$$

is the set of all polynomials. On $\mathcal{R}_k$, we take identification topology (e.g., see [19,p.120]) with respect to the bijections $f_k: \mathbb{R}^k \longrightarrow \mathcal{R}_k$ defined by

$$f_k(a_1, a_2, \cdots, a_k) = (a_1, a_2, \cdots, a_k, 0, \cdots).$$

That is, a set

$$U = \{(a_1, \cdots, a_k, 0, \cdots) \mid (a_1, \cdots, a_k) \in V\}$$

is open in $\mathcal{R}_k$ if and only if V is open in $\mathbb{R}^k$. On $\mathcal{R}$ we impose the inductive limit topology [19,p.420] with respect to the $\mathcal{R}_k$ -- i.e. we impose on $\mathcal{R}$ the finest topology which makes the natural imbeddings $\mathcal{R}_k \subset \mathcal{R}$

continuous. It is routine to prove that $P_k \to P$ in the sense of Definition 2.1 if and only if P_k converges to P in $\mathcal{R}$.

4) It is shown in [6,Lemma 4.3] that, if $\{P_k\}$ is convergent in $\mathbb{R}[s]$, then there exist convergent real sequences $\{\alpha_{ik}\}$, $\{\beta_{ik}\}$, and $\{\gamma_k\}$, with $\alpha_{ik} \to 0$ and $\lim \gamma_k \neq 0$, such that

$$P_k(s) = \gamma_k \prod_i (\alpha_{ik}s-1) \prod_i (s-\beta_{ik}). \tag{3}$$

In particular, if the roots of P_k are bounded, then $\deg P_k = \deg \lim P_k$.

In order to define convergence in $\mathbb{R}(s)$, we adopt a standard quotient space construction over $\mathbb{R}[s]\times(\mathbb{R}[s]-\{0\})$ (e.g., see [18], p.136) and identify each rational function with a unique equivalence class under the relation

$$(a,b)\approx(c,d) \iff ad=bc.$$

We use the expression a/b to denote both a rational function as well as its corresponding equivalence class in $\mathbb{R}[s]\times(\mathbb{R}[s]-\{0\})$. Note that a similar construction may be employed in identifying rational matrices $N/d\in\mathbb{R}^{rm}(s)$ with equivalence classes of pairs $(N,d)\in\mathbb{R}[s]^{rm}\times(\mathbb{R}[s]-\{0\})$.

Adopting ordinary quotient set topology on $\mathbb{R}(s)$, we arrive at the following definition.

__Definition 2.2__ Suppose h_k: $k=1,2,\ldots$, and h are in $\mathbb{R}(s)$. We say __h_k converges to h__ in $\mathbb{R}(s)$, if there exist $n_k \to n$ and $d_k \to d$ in $\mathbb{R}[s]$, with $d_k, d \neq 0$, such that $n_k/d_k=h_k$: $k=1,2,\ldots$, and $n/d=h$.

Along similar lines, we now give three alternative definitions for convergence in $\mathbb{R}(s)^{rm}$.

__Definition 2.3__ Suppose H_k: $k=1,2,\ldots$, and H are $r\times m$ rational matrices with components h_{ijk} and h_{ij}, respectively. We say __H_k converges to H__ in $\mathbb{R}(s)^{rm}$ if $h_{ijk} \to h_{ij}$ in $\mathbb{R}(s)$ as $k \to \infty$.

Definition 2.3' Suppose $H_k : k = 1,2,\cdots$ and H are $r \times m$ rational matrices. We say <u>H_k converges to H</u> in $\mathbb{R}(s)^{rm}$ if there exist $N_k \to N$ in $\mathbb{R}[s]^{rm}$ and $d_k \to d$ in $\mathbb{R}[s]$ such that $N_k/d_k = H_k : k = 1,2,\ldots,$ and $N/d = H$. (Here we assume product topology on $\mathbb{R}[s]^{rm}$ and that the quotient space constructions above are applied componentwise on $\mathbb{R}[s]^{rm} \times (\mathbb{R}[s] - \{0\})$.)

<u>Definition 2.3''</u> We say <u>H_k converges</u> to H in $\mathbb{R}(s)^{rm}$ if there exist $N_k \to N$ in $\mathbb{R}[s]^{rm}$ and $D_k \to D$ in $\mathbb{R}[s]^{m^2}$ with D_k and D nonsingular such that $N_k D_k^{-1} = H_k : k = 1,2,\ldots$ and $ND^{-1} = H$.

<u>Remarks</u>

1) It is easy to show that Definitions 2.3, 2.3' and 2.3'' are equivalent. A fourth alternative definition is the same as 2.3'' except using left instead of right factorizations.

2) Note that a sequence which converges in the sense of Definition 2.3 also converges in identification topology with respect to the map $\mathcal{H} : \mathbb{R}^{n(2n+m+r)} \to \mathbb{R}(s)^{rm}$ defined by

$$\mathcal{H}(E,A,B,C) = \frac{C \cdot \mathrm{adj}(sE - A)B}{\det(sE - A)},$$

where $(E,A,B,C) \in \mathbb{R}^{n(2n+m+r)}$. The construction of the topology on $\mathbb{R}(s)^{rm}$ shows that $\mathcal{H}$ is continuous.

3) If $H_k \to H$ and $G_k \to G$, then $H_k + G_k \to H + G$ and $H_k G_k \to HG$; more generally, $\mathbb{R}(s)^{rm}$ is a topological ring with respect to identification topology on $\mathbb{R}(s)$ and the corresponding product topology on $\mathbb{R}(s)^{rm}$. In particular, relative topology on the subgroup of polynomial matrices $\mathbb{R}[s]^{rm}$ is the same as product topology with respect to Definition 2.1. Note that $\mathbb{R}[s]^{rm}$ is closed in $\mathbb{R}(s)^{rm}$.

We will show in Section 5 that our definition of convergence is the "right" definition for the realization problem, since a sequence in $\mathbb{R}(s)^{rm}$ converges in our sense if and only if it admits a convergent sequence of

state-space realizations. One view of the results of this paper is that they characterize local properties of the map $\mathcal{H}$.

3. Time-Scale Decomposition of Transfer Matrix Sequences

Clearly, any rational matrix H can be uniquely expressed as $H=H_s+H_f$, where H_s is strictly proper and H_f is a polynomial matrix. We now generalize the decomposition to the sequential case: this must be carried out in a way that preserves convergence.

Definition 3.1 1) We say a convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{rm}$ is a slow sequence, if H_k is strictly proper for every k and there exists a bounded region $\Lambda\subset\mathbb{C}$ such that all poles of each H_k lie in Λ.

2) A convergent sequence $\{H_k\}$ is called a fast sequence if for every $M<\infty$ there exists a $K<\infty$ such that $k>K$ implies that each pole p of H_k satisfies $|p|>M$ (all poles tend to infinity).

Remarks

1) The set of all slow sequences in $\mathbb{R}(s)^{rm}$ forms a proper subspace of the real vector space of all convergent sequences in $\mathbb{R}(s)^{rm}$. The same statement holds for fast sequences.

2) Any slow sequence can be expressed as $H_{sk} = N_k/d_k$ where d_k is convergent and monic for every k and $\deg N_k<\deg d_k$, where $\deg N=\max_{i,j}\{\deg n_{ij}\}$ for any polynomial matrix N. Thus $\deg \lim N_k<\deg \lim d_k$. This shows that the limit of every slow sequence is strictly proper.

3) Since the limit of any fast sequence can have no finite poles, such a limit must be a polynomial matrix.

4) Every convergent sequence of polynomial matrices is a fast sequence.

5) If a sequence is both slow and fast, it must be strictly and have no poles whatsoever for large k; hence, the sequence must be identically zero for large k.

6) A sequence of matrices $\{H_k\}$ is slow (fast) if and only if each component sequence $\{h_{ijk}\}$ is slow (fast).

<u>Definition 3.2</u> 1) We say $H_k=H_{sk}+H_{fk}$ is a <u>time-scale decomposition</u> of $\{H_k\}$ when $\{H_{sk}\}$ and $\{H_{fk}\}$ are slow and fast sequences, respectively.

2) In a time-scale decomposition, $\{H_{sk}\}$ and $\{H_{fk}\}$ are called the <u>slow part</u> and the <u>fast part</u> of $\{H_k\}$.

Note that, from remarks 2) and 3) above, if $H_k=H_{sk}+H_{fk}$ is a time-scale decomposition of $\{H_k\}$, then $H_{sk}\to H_s$ and $H_{fk}\to H_f$, where H_s and H_f are the strictly proper part and the polynomial part of $H=\lim H_k$. Theorem 3.3 tells us that every convergent sequence $\{H_k\}$ has an essentially unique time-scale decomposition.

<u>Theorem 3.3</u> 1) For every convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{rm}$, there exist a slow sequence $\{H_{sk}\}$ and a fast sequence $\{H_{fk}\}$ such that $H_k=H_{sk}+H_{fk}$ for every k.

2) If $\{\hat{H}_{sk}\}$ and $\{\hat{H}_{fk}\}$ are slow and fast sequences, respectively, and $H_k=\hat{H}_{sk}+\hat{H}_{fk}$ for every k, then $H_{sk}=\hat{H}_{sk}$ and $H_{fk}=\hat{H}_{fk}$ for sufficiently large k.

<u>Proof</u> 1) We need only treat the case r=m=1; the multivariable case can then be handled componentwise. If $h_k\to h\in\mathbb{R}(s)$, we can find $n_k\to n$ and $d_k\to d$, with $n_k/d_k=h_k$ and $n/d=h$. Since $d_k\to d$, from (3) we can write $d_k=\gamma_k d_{sk}d_{fk}$, where

$$d_{sk}(s) = s^{\mu} + b_{\mu-1,k}s^{\mu-1} + \ldots + b_{0k}$$
$$d_{fk}(s) = a_{\nu k}s^{\nu} + a_{\nu-1,k}s^{\nu-1} + \ldots + a_{1k}s + 1,$$

with each $\{b_{ik}\}$ convergent, $\gamma_k\to\gamma\neq 0$, and $a_{ik}\to 0$ as $k\to\infty$. Let

$$n_k(s) = z_{pk}s^{p} + \ldots + z_{1k}s + z_{0k}.$$

and let $q=\max\{\nu, p-\mu\}$. We will show that there exist convergent polynomial sequences

$$n_{sk} = x_{\mu-1,k}s^{\mu-1} + \ldots + x_{1k}s + x_{0k}$$
$$n_{fk} = y_{q-1,k}s^{q-1} + \ldots + y_{1k}s + y_{0k}$$

such that $n_k/d_k=n_{sk}/d_{sk}+n_{fk}/d_{fk}$. Equivalently, we need to show that

$$\gamma_k(n_{sk}d_{fk} + n_{fk}d_{sk}) = n_k. \tag{4}$$

Note that equation (4) may be written in matrix form

$$\begin{bmatrix} \mathcal{A}_{1k} & \mathcal{B}_{1k} \\ \mathcal{A}_{2k} & \mathcal{B}_{2k} \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = z_k, \tag{5}$$

where

$$\begin{bmatrix} \mathcal{A}_{1k} \\ \mathcal{A}_{2k} \end{bmatrix} = \gamma_k \begin{bmatrix} 1 & & \\ a_{1k} & \ddots & \\ \vdots & \ddots & 1 \\ \vdots & & a_{1k} \\ \vdots & & \vdots \\ a_{\nu k} & & \vdots \\ & \ddots & \vdots \\ & & a_{\nu k} \\ 0 & \cdots & 0 \end{bmatrix}, \quad \begin{bmatrix} \mathcal{B}_{1k} \\ \mathcal{B}_{2k} \end{bmatrix} = \gamma_k \begin{bmatrix} b_{0k} & \cdots & b_{0k} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ b_{\mu-1,k} & \ddots & \vdots \\ 1 & \ddots & b_{\mu-1,k} \\ & & 1 \end{bmatrix}$$

$$x_k = \begin{bmatrix} x_{0k} \\ \vdots \\ x_{\mu-1,k} \end{bmatrix}, \quad y_k = \begin{bmatrix} y_{0k} \\ \vdots \\ y_{q-1,k} \end{bmatrix}, \quad z_k = \begin{bmatrix} z_{0k} \\ \vdots \\ z_{pk} \\ 0 \end{bmatrix},$$

with $\mathcal{A}_{1k}$, $\mathcal{A}_{2k}$, $\mathcal{B}_{1k}$, $\mathcal{B}_{2k}$, and z_k having dimensions $\mu\times\mu$, $q\times\mu$, $\mu\times q$, $q\times q$, and $(\mu+q)\times 1$, respectively. Also note that $\mathcal{A}_{1k}\to\gamma I_\mu$, $\mathcal{B}_{2k}\to\gamma I_\nu - M$, where M is nilpotent and upper triangular, and $\mathcal{A}_{2k}\to 0$ as $k\to\infty$. Hence, there exists a $K<\infty$ such that (5) has a unique solution when $k>K$. For $k\le K$, let h_{sk} be any strictly proper rational function and let $h_{fk}=h_k-h_{sk}$.

2) We have $H_{sk}+H_{fk}=\hat{H}_{sk}+\hat{H}_{fk}$ for sufficiently large k. Hence,

$$H_{sk}-\hat{H}_{sk}=\hat{H}_{fk}-H_{fk}. \tag{6}$$

But the left side of (6) is slow, and the right side is fast. Hence, both sides are identically zero for large k. $\square$

To conclude this section, we note that a time scale decomposition of any transfer matrix sequence of the form $H_k = C_k(sE_k - A_k)^{-1}B_k$, where $\{(E_k, A_k, B_k, C_k)\}$ is convergent in $\mathbb{R}^{n(2n+m+r)}$ and $\det(s \cdot \lim E_k - \lim A_k) \not\equiv 0$, can be achieved by invoking the perturbational form of the classical Weierstrass decomposition for matrix pencils as developed in [7]. Indeed, from [7,p.147], there exist convergent nonsingular matrix sequences $\{M_k\}$ and $\{N_k\}$, with $\lim M_k$ and $\lim N_k$ nonsingular, such that

$$M_k E_k N_k = \begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix}, \quad M_k A_k N_k = \begin{bmatrix} A_{sk} & 0 \\ 0 & I \end{bmatrix}, \tag{7}$$

where $\lim A_{fk}$ is nilpotent. Let

$$\begin{bmatrix} B_{sk} \\ B_{fk} \end{bmatrix} = M_k B_k, \quad [C_{sk} \quad C_{fk}] = C_k N_k. \tag{8}$$

Then

$$H_k(s) = C_{sk}(sI - A_{sk})^{-1}B_{sk} + C_{fk}(sA_{fk} - I)^{-1}B_{fk}. \tag{9}$$

From Definition 3.1 it is clear that

$$H_{sk}(s) = C_{sk}(sI - A_{sk})^{-1}B_{sk} \tag{10}$$

and

$$H_{fk}(s) = C_{fk}(sA_{fk} - I)^{-1}B_{fk} \tag{11}$$

are slow and fast sequences, respectively. Hence, (9) is a time-scale decomposition of $\{H_k\}$.

4. The Characteristic Polynomial Sequence

In this section we investigate several useful properties of the sequence of characteristic polynomials corresponding to a convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{rm}$. We first extend the conventional definition of the characteristic polynomial to improper transfer matrices. Recall that the characteristic polynomial Δ of a strictly proper rational matrix H_s is defined as the least common monic denominator of all minors of H_s.

Definition 4.1 If H is a rational matrix with $H=H_s+H_f$ for some strictly proper H_s and polynomial matrix H_f, the characteristic polynomial Δ of H is defined as the characteristic polynomial of H_s.

Consider the sequence of characteristic polynomials $\{\Delta_k\}$ corresponding to $\{H_k\}$. Since $H_k N_k d_k$, it follows that Δ_k divides $d_k^{\min\{r,m\}}$ for each k; thus, boundedness of $\{\deg d_k\}$ ensures boundedness of $\{\deg \Delta_k\}$. Let $\eta=\overline{\lim}\{\deg \Delta_k\}$, and note that $\deg \Delta_k \le \eta$ for sufficiently large k. For all such k, Δ_k can thus be uniquely identified with a point $\langle\Delta_k\rangle$ in the real projective space $\mathbb{P}^\eta$ (see, e.g., [20]) according to

$$s^i + \alpha_{i-1}s^{i-1} + \dots + \alpha_0 \mapsto (0,\dots,0,1,\alpha_{i-1},\dots,\alpha_0) \in \mathbb{R}^{\eta+1}.$$

In fact, there is a one-to-one correspondence between $\mathbb{P}^\eta$ and the set of monic polynomials Δ with $\deg \Delta \le \eta$. These observations lead to the following definition.

Definition 4.2 Let $\{H_k\}$ be any convergent sequence in $\mathbb{R}(s)^{rm}$, and let Δ_k be the characteristic polynomial of H_k. Set

$$\rho_k = \begin{cases} \langle\Delta_k\rangle, & \deg \Delta_k \le \eta \\ \langle 1\rangle, & \deg \Delta_k > \eta \end{cases}.$$

The sequence $\{\rho_k\}$ is called the characteristic polynomial (CP) of $\{H_k\}$.

It is easy to show that $\{\rho_k\}$ converges if and only if there exists a real sequence $\{\gamma_k\}$ such that $\{\gamma_k\Delta_k\}$ converges to a non-zero limit $\Delta\in\mathbb{R}[s]$. In this case, $\lim \rho_k=\langle\Delta\rangle$. We now present several pathological situations that can arise in dealing with the CP.

Example 4.3

1) The following example illustrates that when $\{H_k\}$ is convergent, the corresponding CP may not converge. Consider the sequence

$$H_k(s) = \begin{cases} \dfrac{(s+2)}{(s+1)(s-2-\frac{1}{k})}, & k \text{ even} \\[2ex] \dfrac{(s-3)}{(s+1)(s+3+\frac{1}{k})}, & k \text{ odd} \end{cases}.$$

and let $H(s)=\frac{1}{s-1}$ We may write $H_k = N_k\ d_k$, where $N_k=(s+2)(s-3)$ and

$$d_k = \begin{cases} (s-1)(s-2-\frac{1}{k})(s+3), & k \text{ even} \\ (s-1)(s+2)(s+3-\frac{1}{k}), & k \text{ odd} \end{cases}.$$

Thus $H_k \longrightarrow H$, but

$$\Delta_k(s) = \begin{cases} (s+1)(s-2+\frac{1}{k}), & k \text{ even} \\ (s-1)(s-3+\frac{1}{k}), & k \text{ odd} \end{cases}.$$

Clearly, $\{\rho_k\}$ is not convergent. Note, however, that $\{H_k\}$ can be divided into two subsequences with convergent CP's according to $H_k^{(1)}=H_{2k-1}$ and $H_k^{(2)}=H_{2k}$.

2) In some cases, $\{\rho_k\}$ may converge even though $\{\Delta_k\}$ does not. Consider the rational sequence

$$H_k(s) = \frac{1}{(\frac{1}{k}s-1)(s+2)}.$$

In this case, $\{H_k\}$ has CP determined by $\Delta_k(s)=(s+k)(s+2)$, but $\{\rho_k\}$ converges, since

$$\frac{1}{k}\Delta_k(s) = \frac{1}{k}(s+k)(s+2) \longrightarrow s+2.$$

3) Finally, we note that convergence of $\{\rho_k\}$ (or even $\{\Delta_k\}$) does not guarantee that $\lim \Delta_k$ is the characteristic polynomial of $\lim H_k$. For example, let

$$H_k(s) = \frac{s+2}{(s-1)(s-2+\frac{1}{k})}.$$

Then

$$\Delta_k(s) = (s+1)(s+2+\frac{1}{k}) \longrightarrow (s-1)(s-2).$$

But

$$H_k(s) \longrightarrow \frac{1}{s-1}.$$

Next we examine some basic properties of the CP with respect to the time-scale decomposition. First we need a simple result for individual systems.

Lemma 4.4 Suppose $H=H_1-H_2$, where H_1 and H_2 have no common poles, and let Δ, Δ_1, and Δ_2 be the characteristic polynomials of H, H_1, and H_2, respectively Then $\Delta=\Delta_1\Delta_2$.

Proof From the definition of the CP we can assume without loss of generality that H_1 and H_2 are strictly proper. Suppose (A_1,B_1,C_1) and (A_2,B_2,C_2) are minimal realizations of H_1 and H_2; then $\Delta_i(s)=\det(sI-A_i)$. If we let

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ C = [C_1 \ C_2],$$

then (A,B,C) is a minimal realization of H with CP

$$\Delta(s) = \det(sI-A_1)\det(sI-A_2).$$

In particular, for any time-scale decomposition, Lemma 4.4 implies that, when k is sufficiently large, we have

$$\Delta_k = \Delta_{sk}\Delta_{fk}, \tag{12}$$

where Δ_k, Δ_{sk}, and Δ_{fk} are the characteristic polynomials of H_k, H_{sk}, and H_{fk}, respectively.

Lemma 4.5 Let $H_k=H_{sk}+H_{fk}$ be a time-scale decomposition.

1) If Δ_{sk} is the characteristic polynomial of H_{sk} and $\Delta\in\mathbb{R}[s]$ is monic, then $\langle\Delta_{sk}\rangle\longrightarrow\langle\Delta\rangle$ if and only if $\Delta_{sk}\longrightarrow\Delta$.

2) If $\{\rho_{fk}\}$ is the CP of H_{fk}, then $\rho_{fk}\longrightarrow\langle 1\rangle$.

3) The CP of $\{H_k\}$ is convergent if and only if the CP of $\{H_{sk}\}$ is convergent. When the two CP's converge, their limits coincide.

Proof 1) Sufficiency is obvious. To show necessity, observe that there must exist a real sequence $\{\gamma_k\}$ such that $\gamma_k \Delta_{sk} \longrightarrow \Delta$ and that $\{\Delta_{sk}\}$ has bounded roots. From (3), $\{\gamma_k\}$ converges. Since Δ_{sk} and Δ are monic, $\gamma_k \equiv 1$

2) We have $\rho_{fk} = \langle \Delta_{fk} \rangle$, where

$$\Delta_{fk} = \begin{cases} \prod_i (s+\lambda_{ik}), & H_{fk} \text{ is not a polynomial matrix.} \\ 1, & H_{fk} \text{ is a polynomial matrix.} \end{cases}$$

Here the λ_{ik} satisfy the property that, for every $M<\infty$, there exists a $K<\infty$ such that $|\lambda_{ik}| > M$ for each i and each $k>K$. Let

$$\gamma_k = \begin{cases} \prod_i \frac{1}{\lambda_{ik}}, & H_{fk} \text{ is not a polynomial matrix.} \\ 1, & H_{fk} \text{ is a polynomial matrix.} \end{cases}$$

Then $\gamma_k \Delta_{fk} \longrightarrow 1$ in $\mathbb{R}[s]$.

3) The result follows immediately from (12) and part 2). □

The final result of this section focuses on the observation made in Example 4.3, part 1), that the CP of a sequence $\{H_k\}$ in $\mathbb{R}^{rm}(s)$ which is not convergent can sometimes be decomposed into convergent subsequences. We can in fact demonstrate that a finite decomposition of this sort can always be achieved.

Theorem 4.6 If $H_k \longrightarrow H$, then $\{\rho_k\}$ has finitely many limit points.

Proof From Lemma 4.5, part 3), we need only consider the case where $\{H_k\}$ is a slow sequence. Let $H_k = N_k/d_k$ and $H=N/d$. From the definition of the characteristic polynomial, Δ_k divides $d_k^{\min\{r,m\}}$ for each k. But $d_k^{\min\{r,m\}} \longrightarrow d^{\min\{r,m\}}$, so the unique monic representative of each limit point of $\{\rho_k\}$ must divide $d^{\min\{r,m\}}$. The result then follows from the fact that any polynomial over $\mathbb{R}$ has finitely many monic divisors. □

Corollary 4.7 If $H_k \to H$, then there exist finitely many strictly increasing sequences $\{k_j^i\}: i=1,\dots,\pi$ of positive integers such that

1) $\{k_j^i \mid i=1,2,\dots,\pi; j=1,2,\dots\} = \{1,2,3,\dots\}$,

2) $\{k_j^p \mid j=1,2,\dots\} \cap \{k_j^q \mid j=1,2,\dots\} = \phi$ when $p \neq q$.

3) each $\{H_{k_j^i}\}$ has convergent CP.

Proof From Theorem 4.6, there are only finitely many limit points $\rho^1,\dots,\rho^\pi \in \mathbb{P}^\eta$ of $\{\rho_k\}$. Since $\mathbb{P}^\eta$ is a compact Hausdorff space, each open subset U of $\mathbb{P}^\eta$ satisfying $\{\rho^1,\dots,\rho^\pi\} \subset U$ contains a tail of $\{\rho_k\}$. Indeed, otherwise there would exist a subsequence of $\{\rho_k\}$ with no limit point, contradicting compactness of $\mathbb{P}^\eta$. Let $U_1,\dots,U_\pi$ be nonintersecting neighborhoods of $\rho^1,\dots,\rho^\pi$, respectively; then there exists a $K<\infty$ such that $\{\rho_k\} \subset \cup U_i$ for $k>K$. Let $k_j^1=j$: $j=1,\dots,K$. The remaining k_j^i may then be defined iteratively according to $k_j^i=\min(\{k \mid \rho_k \in U_i\} - \{k_q^i \mid q<j\})$. If $V_i \subset U_i$ is another neighborhood of ρ_i, then by compactness of $\mathbb{P}^\eta$ there must be a tail of the subsequence $\{\rho_{k_j^i}\}$ contained in V_i; hence, $\rho_{k_j^i} \to \rho^i$.

5. Existence of Realizations

We are now ready to formally define realizations of a given transfer matrix sequence $\{H_k\}$ and discuss their existence. We base our definition of a realization of $\{H_k\}$ on the standard definition of a realization of a single rational matrix H as in Theorem 1.2.

Definition 5.1 1) Suppose $\{H_k\}$ converges in $\mathbb{R}(s)^{rm}$. We say a sequence $\{\sigma_k\}$ in $\mathbb{R}^{n(2n+m+r)}$ is a realization of $\{H_k\}$, if there exists an integer K and a $\sigma\in\mathbb{R}^{n(n+m+r)}$ such that σ_k is a realization of H_k when $k>K$ and $\sigma_k \longrightarrow \sigma$ in $\mathbb{R}^{n(2n+m+r)}$

2) A realization $\{\sigma_k\}$ in $\mathbb{R}^{n(n+m+r)}$ is said to have dimension n.

Note that the dimension of a realization $\{\sigma_k\}$ is given simply by $\dim \sigma_k$ for any k. If $H_k \longrightarrow H$, then continuity of $\mathcal{H}$ implies that σ is a realization of H in the conventional sense. We will show that there exists a realization for any convergent sequence $\{H_k\}$; this generalizes part 1) of Theorem 1.2 to sequences and demonstrates that the definition of convergence in $\mathbb{R}(s)^{rm}$ outlined in Section 2 is the correct one for our purposes.

To simplify subsequent discussion, we will make use of the mapping $\mathcal{G}:\mathbb{R}(s)^{rm} \longrightarrow \mathbb{R}(s)^{rm}$ defined by

$$\mathcal{G}(H)(s) = \frac{1}{s}H(\frac{1}{s}).$$

It is easy to see that $\mathcal{G}$ is an isomorphism on $\mathbb{R}(s)^{rm}$ and that $\mathcal{G}^{-1}=\mathcal{G}$. Some elementary properties of $\mathcal{G}$ follow.

Lemma 5.2 Let $H\in\mathbb{R}(s)^{rm}$, and let $\{H_k\}$ be convergent in $\mathbb{R}(s)^{rm}$.

1) (E,A,B,C) is a realization of H if and only if (A,E,B,C) is a realization of $\mathcal{G}(H)$.

2) $\mu(H)=\mu(\mathcal{G}(H))$

3) If the characteristic polynomial of H is $\Delta(s)=s^n+\eta_{n-1}s^{n-1}+\ldots+\eta_0$, then the characteristic polynomial of $\mathcal{G}(H)$ is $\tau(s)=\gamma(\eta_0 s^n+\ldots+\eta_{n-1}s+1)$ for some $\gamma\neq 0$.

Proof 1) Suppose $H(s)=C(sE-A)^{-1}B$. Then

$$\mathcal{G}(H)(s) = \frac{1}{s}C(\frac{1}{s}E-A)^{-1}B = C(sA-E)^{-1}B.$$

2) From 1), if H has a realization of degree n, then so does $\mathfrak{I}(H)$. The converse follows from $\mathfrak{I}(\mathfrak{I}(H))=H$.

3) Let (E,A,B,C) be a minimal realization of H; then (A,E,B,C) is a minimal realization of $\mathfrak{I}(H)$. Hence, $\Delta(s)=\gamma_1 \det(sE-A)$ for some $\gamma_1 \neq 0$ (for details, see [14]), and

$$\begin{aligned}
\tau(s) &= \gamma_2 \det(sA-E) \\
&= \gamma_2(-s)^n \det(\tfrac{1}{s}E-A) \\
&= \frac{\gamma_2}{\gamma_1}(-s)^n\left((\tfrac{1}{s})^n+\eta_{n-1}(\tfrac{1}{s})^{n-1}+\dots+\eta_0\right) \\
&= (-1)^n \cdot \frac{\gamma_2}{\gamma_1}(\eta_0 s^n+\dots+\eta_{n-1}s+1).
\end{aligned}$$

Lemma 5.3 If $\{H_k\}$ is a fast sequence, then $\{\mathfrak{I}(H_k)\}$ is a slow sequence.

Proof Since all poles of H_k tend to infinity, we can write

$$H_k(s) = \frac{N_k(s)}{\prod_{i=1}^{p}(\alpha_{ik}s-1)},$$

where $\{N_k\}$ is convergent and each $\alpha_{ik} \to 0$ as $k \to \infty$. Let $q=\max_k\{\deg n_{ijk}, p-1\}$. Then

$$\mathfrak{I}(H_k)(s) = (-1)^{p-1}\frac{s^q N_k(\frac{1}{s})}{s^{q-p+1}\prod(s-\alpha_{ik})}.$$

Note that $s^q N_k(\frac{1}{s})$ is a polynomial matrix, each of whose elements has degree at most q. Clearly, $\mathfrak{I}(H_k)$ has bounded poles, and the denominator of $\mathfrak{I}(H_k)$ has degree $q+1$, so $\mathfrak{I}(H_k)$ is strictly proper.

Now consider a time-scale decomposition $H_k=H_{sk}+H_{fk}$ of an arbitrary $\{H_k\}$ in $\mathbb{R}(s)^{rm}$. Suppose $\{H_{sk}\}$ and $\{\mathfrak{I}(H_{fk})\}$ have realizations of the form $\{(I,A_{sk},B_{sk},C_{sk})\}$ and $\{(I,A_{fk},B_{fk},C_{fk})\}$. Then each $H_{fk}=\mathfrak{I}(\mathfrak{I}(H_{fk}))$ has (A_{fk},I,B_{fk},C_{fk}) as one of its realizations. Defining

$$E_k = \begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix}, \quad A_k = \begin{bmatrix} A_{sk} & 0 \\ 0 & I \end{bmatrix}, \quad B_k = \begin{bmatrix} B_{sk} \\ B_{fk} \end{bmatrix}, \quad C_k = [C_{sk} \ C_{fk}],$$

it is easy to check that $\{(E_k, A_k, B_k, C_k)\}$ is a realization of $\{H_k\}$. Therefore, we only need to prove existence of realizations for slow sequences.

Theorem 5.4 Every slow sequence has a realization of the form $\{(I, A_k, B_k, C_k)\}$.

Proof First we treat the case r=m=1. Let $H_k = N_k/d_k$, where $\{N_k\}$ and $\{d_k\}$ are convergent in $\mathbb{R}[s]$. Then

$$d_k = s^q + \alpha_{q-1,k} s^{q-1} + \cdots + \alpha_{0k}$$

$$N_k = \beta_{q-1,k} s^{q-1} + \beta_{q-2,k} s^{q-2} + \cdots + \beta_{0k},$$

where α_{ik} and β_{ik} converge as $k \to \infty$. To obtain a realization of $\{H_k\}$ of the desired form, set

$$A_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\alpha_{0k} & -\alpha_{1k} & \cdots & -\alpha_{q-1,k} \end{bmatrix}, \quad B_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_k = [\beta_{0k} \ \ \beta_{1k} \ \cdots \ \beta_{q-1,k}].$$

Now we consider the general case. By the definition of convergence in $\mathbb{R}(s)^{rm}$, every component sequence $\{h_{ijk}\}$ is convergent. Suppose $\{(I, A_k^{ij}, B_k^{ij}, C_k^{ij})\}$ is a realization of $\{h_{ijk}\}$. Let

$$A_k^i = \mathrm{diag}\{A_k^{ij} \mid j=1,2,\ldots,m\}, \quad B_k^i = \mathrm{diag}\{B_k^{ij} \mid j=1,2,\cdots,m\}, \quad C_k^i = [C_k^{i1} \cdots C_k^{im}]$$

and

$$A_k = \mathrm{diag}\{A_k^i \mid i=1,2,\ldots,r\}, \quad B_k = \begin{bmatrix} B_k^1 \\ \vdots \\ B_k^r \end{bmatrix}, \quad C_k = \mathrm{diag}\{C_k^i \mid i=1,2,\cdots,r\}.$$

A simple calculation verifies that $\{(I, A_k, B_k, C_k)\}$ is a realization of $\{H_k\}$. $\square$

Combining the time-scale decomposition with Lemma 5.3 and Theorem 5.4, we arrive at the following result.

Corollary 5.5 Every convergent sequence in $\mathbb{R}(s)^{rm}$ has a realization.

Theorem 5.4 (but not Corollary 5.5) may also be proven in an abstract algebraic framework as outlined in [17,Chapter 4]. Briefly, consider the commutative ring c of convergent sequences in $\mathbb{R}$ using pointwise operations, and let the set of r×m proper rational real matrices be denoted by $\mathbb{R}_p(s)^{rm}$. A convergent sequence $\{H_k\}$ in $\mathbb{R}_p(s)^{rm}$ may then be viewed as a formal power series over the ring of r×m matrices with elements in c. Indeed, we may expand each element of each H_k about s=∞, yielding the series

$$H_k = \sum_{i=1}^{\infty} (\frac{1}{s})^i H_{ik}, \tag{13}$$

where the sequences $\{H_{ik}\}$ are convergent. From this point there are two ways to proceed. First, one can prove realizability by constructing a certain infinite-dimensional Hankel matrix from the H_{ik}. It must then be shown that the span of the columns of the Hankel matrix is a finitely generated module over c. A second approach is to show that the formal power series (13) is "rational" in a certain algebraic sense. This immediately guarantees realizability. Both conditions can be demonstrated in our framework fairly easily; however, our proof of Theorem 5.4 is more direct and is sufficient for our purposes.

6. Minimality

In Section 5 we showed that every convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{rm}$ has a convergent realization $\{\sigma_k\}$. In this section, we explore the issue of minimality of a realization.

Definition 6.1 1) If n is the smallest integer such that $\{H_k\}$ has a realization of dimension n, and $\{\sigma_k\}$ is a realization of $\{H_k\}$ with $\dim \sigma_k = n$, then we say $\{\sigma_k\}$ is a minimal realization of $\{H_k\}$.

2) If a sequence of state-space systems $\{\sigma_k\}$ is a minimal realization of its transfer matrix sequence, we say $\{\sigma_k\}$ is minimal.

Obviously, all minimal realizations of $\{H_k\}$ have the same dimension. This fact enables us to define a degree function δ on the set of convergent rational matrix sequences by setting $\delta\{H_k\}$ equal to the dimension of any minimal realization of $\{H_k\}$. In this section we develop a simple expression for $\delta\{H_k\}$ for slow sequences and then extend it to the general case. Next, we examine a natural conjecture for determining whether a sequence $\{\sigma_k\}$ is minimal and relate minimal realizations of the same $\{H_k\}$ in a manner analogous to Theorem 1.2, part 4). Finally, we relate our results to the realization theory outlined in [17] for algebraic systems over the ring c.

In our development it will be helpful to exploit various properties of the mapping which takes each state-space system into a particular choice of numerator and denominator of its transfer function. Specifically, define $\Gamma_n : \mathbb{R}^{n(n+m+r)} \to \mathbb{R}^{n(rm+1)+1}$ according to

$$\Gamma_n(A,B,C) = (C\cdot \mathrm{adj}(sI-A)B, \det(sI-A)).$$

Here we have identified $\mathcal{R}_k$, as defined in Remark 3) after Definition 2.1, with $\mathbb{R}^k$. Note that Γ_n is continuous; if $\Gamma_n(A,B,C)=(N,d)$, then (I,A,B,C) is a realization of N/d. Also notice the distinction between Γ_n and $\mathcal{H}$, as defined in Section 2. We denote $\mathrm{Im}\,\Gamma_n = \Gamma_n(\mathbb{R}^{n(n+m+r)})$.

The following series of lemmas leads us to the first main theorem of this section.

Lemma 6.2 Consider any pair (N,d) where d is monic, $\deg d = n$, and N/d is strictly proper with characteristic polynomial Δ. Then $(N,d) \in \mathrm{Im}\,\Gamma_n$ if and only if Δ divides d.

Proof (Sufficient) Suppose

$$d(s) = \Delta(s) \prod_{i=1}^{p} (s-\beta_i),$$

and let (I,A,B,C) be a minimal realization of N/d. Define

$$A^* = \begin{bmatrix} A & 0 \\ 0 & \Sigma \end{bmatrix}, \quad B^* = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C^* = [C \quad 0],$$

where

$$\Sigma = \begin{bmatrix} -\beta_1 & & & \\ & -\beta_2 & & \\ & & \ddots & \\ & & & -\beta_p \end{bmatrix}.$$

Then

$$\det(sI-A^*) = \det(sI-A)\det(sI-\Sigma) = \Delta(s) \prod_{i=1}^{p} (s+\beta_i) = d(s).$$

Since $C^*(sI-A^*)^{-1}B^* = C(sI-A)^{-1}B = N(s)/d(s)$,

$$C^* \operatorname{adj}(sI-A^*)B^* = \det(sI-A^*)\frac{N(s)}{d(s)} = N(s).$$

Hence, $(N,d) = \Gamma_n(A^*,B^*,C^*)$.

(Necessary) Suppose $(N,d) = \Gamma_n(A,B,C)$; then (I,A,B,C) is a realization of N/d. From [16, Theorem 5-18], we can find a similarity transformation T such that

$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \quad CT = [0 \quad C_2 \quad C_3],$$

where (A_{22},B_2,C_2) is a minimal realization of N/d. Note that $\det(sI-A_{22}) = \Delta(s)$. Thus

$$d(s) = \det(sI-A) = \det(sI-A_{11})\det(sI-A_{33})\Delta(s).$$

From Corollary 4.7, $\{H_k\}$ can be decomposed into π sequences $\{H_k^{(i)}\}; i=1,2,\ldots,\pi$, where we define $H_j^{(i)} = H_{k_j^i}$. Each sequence has convergent CP satisfying

$$\langle \Delta_k^{(i)} \rangle \longrightarrow \langle \Delta^{(i)} \rangle; \quad i=1,2,\ldots,\pi,$$

where $\Delta^{(i)}$ is monic. If $\{H_k\}$ is slow, then, from Lemma 4.3, part 1), $\Delta_k^{(i)} \to \Delta^{(i)}$. Let

$$\hat{\Delta} = \mathrm{LCM}\,\{\Delta^{(1)},\ldots,\Delta^{(\pi)}\},$$

where LCM denotes the least common multiple. Also define

$$\hat{\Delta}_k^{(i)} = \Delta_k^{(i)} \frac{\hat{\Delta}}{\Delta^{(i)}};\ i=1,2,\ldots,\pi. \tag{14}$$

Note that each $\hat{\Delta}_k^{(i)}$ is a polynomial and that, if $\{H_k\}$ is slow, $\hat{\Delta}_k^{(i)} \to \hat{\Delta}$.

Lemma 6.3 Let $\{H_k\}$ be a slow sequence with $H_k \to H$, and suppose H has characteristic polynomial Δ. Then Δ divides $\hat{\Delta}$.

Proof According to Corollary 4.7, $\{H_k\}$ can be decomposed into π subsequences with convergent CP's $\Delta_k^{(i)} \to \Delta^{(i)}$. If Δ divides $\Delta^{(i)}$ for each i, then Δ divides $\hat{\Delta}$. Hence, it suffices to treat the case where $\{H_k\}$ has convergent CP $\Delta_k \to \hat{\Delta}$.

Let $p=\min\{r,m\}$, and consider the sequence $\{\tilde{H}_k\}$ of $1\times \sum_{i=1}^{p} \binom{r}{i}\binom{m}{i}$ rational matrices, each $\tilde{H}_k$ consisting of the minors of H_k of all orders. Obviously, $\tilde{H}_k \to \tilde{H}$, where $\tilde{H}$ is defined similarly. It follows from elementary arguments that $\tilde{H}_k$ has characteristic polynomial Δ_k (same as H_k) and that, for any polynomial q, $q\tilde{H}_k$ is a polynomial matrix if and only if Δ_k divides q. In particular, $\Delta_k \tilde{H}_k$ is a polynomial matrix. Since $\mathbb{R}[s]^j$ is closed in $\mathbb{R}(s)^j$ for any j, $\hat{\Delta}\tilde{H}$ is a polynomial matrix. Thus, the characteristic polynomial Δ of $\tilde{H}$ (and H) divides $\hat{\Delta}$. $\square$

Lemma 6.4 Γ_n is an open mapping.

Proof Note that Γ_n is multilinear; thus, it is a composition of functions on Euclidean spaces $\mathbb{R}^p$ of the form $f(x_1,\ldots,x_p)=\pm x_i x_j$ and $g(x_1,\ldots,x_p)=x_1+\cdots+x_p$. Since f and g are open, compositions of open maps are open, and products of opens sets are open, it follows that Γ_n is open. $\square$

Lemma 6.5 Let X and Y be topological spaces with X first countable, and let $Q:X\longrightarrow Y$ be an onto, open, and continuous map. For any convergent sequence $\{y_k\}$ in Y with $y_k\longrightarrow y\in Y$ and any $x\in Q^{-1}(y)$, there exist $x_k\in Q^{-1}(y_k):k=1,2,\ldots$ such that $x_k\longrightarrow x$.

Proof Let $\{U_i:i=1,2,\cdots\}$ be a countable basis of neighborhoods of x with $U_i\supset U_{i+1}$. Since Q is open, each $V_i=Q(U_i)$ is a neighborhood of y. Hence, for any V_i, we can find an integer K_i such that $y_k\in V_i$ when $k>K_i$. Furthermore, there must exist points $x_k^{(i)}\in U_i:k=K_i+1,K_i+2,\ldots$ with $Q(x_k^{(i)})=y_k$. For $k\le K_i$, select any $x_n^{(i)}\in Q^{-1}(y_n)$. This process defines sequences $\{x_k^{(i)}\}:i=1,2,\ldots$ Without loss of generality, we may assume $K_{i+1}>K_i$. If we let $x_k=x_k^{(i)}:k=K_{i-1}-1,\ldots,K_i$, where $K_0=0$, the construction shows that each U_i contains a tail of the sequence $\{x_k\}$. Hence, $x_k\longrightarrow x$. ∎

Lemma 6.6 Suppose $\{H_k\}$ is a slow sequence with $H_k\longrightarrow H$. If there are pairs $(N_k,d_k),(N,d)\in \operatorname{Im}\Gamma_n$ such that $N_k/d_k=H_k$, $N/d=H$, and $(N_k,d_k)\longrightarrow(N,d)$, then $\{H_k\}$ has an n-dimensional realization.

Proof Note that $\mathbb{R}^{n(n+m+r)}$ is first countable. Thus, if we restrict the range of Γ_n to $\operatorname{Im}\Gamma_n$, we may use Lemmas 6.4 and 6.5 and the fact that Γ_n is continuous to conclude that there exists a convergent sequence $(A_k,B_k,C_k)\longrightarrow(A,B,C)$ in $\mathbb{R}^{n(n+m+r)}$ such that $\Gamma_n(A_k,B_k,C_k)=(N_k,d_k)$ and $\Gamma_n(A,B,C)=(N,d)$. Notice that (I,A_k,B_k,C_k) is a realization of $N_k/d_k=H_k;k=1,2,\ldots,$ and (I,A,B,C) is a realization of $N/d=H$. ∎

Lemma 6.7 If a slow sequence $\{H_k\}$ has an n-dimensional realization, then it has an n-dimensional realization of the form $\{(I,A_k,B_k,C_k)\}$.

Proof Suppose $\{H_k\}$ has a realization having dimension n. Since $\{H_k\}$ is slow, the decomposition (7)-(11) shows that $\{H_k\}$ is of the form

$$H_k(s) = C_{sk}(sI-A_{sk})^{-1}B_{sk},$$

where A_{sk} is $q \cdot q$ with $q \le n$. Define

$$A_k = \begin{bmatrix} A_{sk} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_k = \begin{bmatrix} B_{sk} \\ 0 \end{bmatrix}, \quad C_k = [C_{sk} \quad 0].$$

Theorem 6.8 For any slow sequence $\{H_k\}$,

$$\delta\{H_k\} = \deg \hat{\Delta}.$$

Proof Let $n_0 = \deg \hat{\Delta}$. We first show that there exists an n_0-dimensional realization of $\{H_k\}$. Let $\hat{N}_k^{(i)} = \hat{\Delta}_k^{(i)} H_k^{(i)}$ and $\hat{N} = \hat{\Delta} H$; then $\hat{N}_k^{(i)}$ and $\hat{N}$ are polynomial matrices with $(\hat{N}_k^{(i)}, \hat{\Delta}_k^{(i)}) \longrightarrow (\hat{N}, \hat{\Delta})$. Since all poles of H_k are bounded, Remark 4) after Definition 2.1 shows that $\deg \hat{\Delta}_k^{(i)} = \deg \hat{\Delta} = n_0$. Thus, from (14) and Lemmas 6.2 and 6.3, $(\hat{N}_k^{(i)}, \hat{\Delta}_k^{(i)}), (\hat{N}, \hat{\Delta}) \in \operatorname{Im} \Gamma_{n_0}$. Suppose $\{(\hat{N}_k, \hat{\Delta}_k)\}$ is constructed by setting $\hat{N}_{k_j^i} = \hat{N}_j^{(i)}$ and $\hat{\Delta}_{k_j^i} = \hat{\Delta}_j^{(i)}$ whenever $H_{k_j^i} = H_j^{(i)}$, where the k_j^i are defined by the decomposition of Corollary 4.7. Then $(\hat{N}_k, \hat{\Delta}_k) \in \operatorname{Im} \Gamma_{n_0}$. The desired result follows from Lemma 6.6.

It remains to show that n_0 is the minimal dimension over all realizations of $\{H_k\}$. Suppose $\{H_k\}$ has an n-dimensional realization. Then, from Lemma 6.7, it has an n-dimensional realization of the form $\{(I, A_k, B_k, C_k)\}$. Let $(N_k, d_k) = \Gamma_n(A_k, B_k, C_k)$ and $(N, d) = \Gamma_n(\lim(A_k, B_k, C_k))$; from Lemma 6.2, $\hat{\Delta}_k$ divides d_k for every k. Letting $d_j^{(i)} = d_{k_j^i}$, $\hat{\Delta}_k^{(i)}$ divides $d_k^{(i)}$. Since Γ_n is continuous, $d_k^{(i)} \longrightarrow d$; thus, closure of $\mathbb{R}[s] \subset \mathbb{R}(s)$ guarantees that each $\hat{\Delta}^{(i)}$ divides d. Thus $\hat{\Delta}$ divides d, and

$$n = \deg d \ge \deg \hat{\Delta} = n_0.$$

The following result offers one method of calculating $\delta\{H_k\}$ for an arbitrary convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{rm}$.

Theorem 6.9 If $H_k = H_{sk} + H_{fk}$ is a time-scale decomposition, then

$$\delta\{H_k\} = \delta\{H_{sk}\} + \delta\{H_{fk}\}.$$

Proof Suppose $\{(E_k, A_k, B_k, C_k)\}$ is a minimal realization of $\{H_k\}$. Appealing to (7)-(11), it suffices to show that $\{\sigma_{sk}\}=\{(I, A_{sk}, B_{sk}, C_{sk})\}$ and $\{\sigma_{fk}\}=\{(A_{fk}, I, B_{fk}, C_{fk})\}$ are minimal realizations of $\{H_{sk}\}$ and $\{H_{fk}\}$. Suppose there exists a realization $\{\bar{\sigma}_{sk}\}=\{(\bar{E}_{sk}, \bar{A}_{sk}, \bar{B}_{sk}, \bar{C}_{sk})\}$ of $\{H_{sk}\}$ with $\dim \bar{\sigma}_{sk} < \dim \sigma_{sk}$. By Lemma 6.7, we may assume that $\bar{E}_{sk}=I$. Let

$$\bar{E}_k = \begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix}, \quad \bar{A}_k = \begin{bmatrix} \bar{A}_{sk} & 0 \\ 0 & I \end{bmatrix}, \quad \bar{B}_k = \begin{bmatrix} \bar{B}_{sk} \\ B_{fk} \end{bmatrix}, \quad \bar{C}_k = [\bar{C}_{sk} \;\; C_{fk}].$$

Then $\{\bar{\sigma}_k\}=\{(\bar{E}_k, \bar{A}_k, \bar{B}_k, \bar{C}_k)\}$ is a realization of $\{H_k\}$ with $\dim \bar{\sigma}_k < \dim \sigma_k$. This is a contradiction. A similar argument shows minimality of $\{\sigma_{fk}\}$. $\square$

Thus, one way to find $\delta\{H_k\}$ is to first perform a time-scale decomposition $H_k = H_{sk} + H_{fk}$ and then to use Theorem 6.8 to find $\delta\{H_{sk}\}$ and $\delta\{H_{fk}\} = \delta\{\mathscr{G}(H_{fk})\}$. Fortunately, the next theorem simplifies this task and shows how to calculate $\delta\{H_k\}$ without resorting to time-scale decomposition. Recall that, for any $H \in \mathbb{R}(s)^{rm}$, H_f denotes the polynomial part of H.

Theorem 6.10 Suppose $H_k \longrightarrow H$. Then

$$\delta\{H_k\} = \max_i \overline{\lim_k} \left(\deg \hat{\Delta}_k^{(i)} + \mu((H_k^{(i)})_f)\right).$$

Proof Suppose $H_k = H_{sk} - H_{fk}$ is a time-scale decomposition of $\{H_k\}$, and let

$$H_{sj}^{(i)} = H_{sk_j^i}, \quad H_{fj}^{(i)} = H_{fk_j^i}.$$

It is clear that $H_k^{(i)} = H_{sk}^{(i)} + H_{fk}^{(i)}$ is also a time-scale decomposition and that $H_{sk}^{(i)}$ and $H_{fk}^{(i)}$ have characteristic polynomials $\Delta_{sk}^{(i)}$ and $\Delta_{fk}^{(i)}$, respectively. From (12),

$$\Delta_k^{(i)} = \Delta_{sk}^{(i)} \Delta_{fk}^{(i)}. \tag{15}$$

From (15) and Lemma 4.5, part 2), $\Delta_{sk}^{(i)} \longrightarrow \Delta^{(i)}$. Hence, from Theorem 6.8, $\delta\{H_{sk}\} = \deg \hat{\Delta}$. Also, since each $\Delta_{sk}^{(i)}$ is convergent and monic, $\deg \Delta_{sk}^{(i)} = \deg \Delta^{(i)}$ for large k.

From (3), $\langle\Delta_{fk}\rangle$ is of the form

$$\langle\Delta_{fk}\rangle = \langle \prod_{i=1}^{p} (\alpha_{ik}s-1)\rangle = \langle \varepsilon_{pk}s^p+\dots+\varepsilon_{1k}s+1\rangle,$$

where $\varepsilon_{ik}\to 0$ as $k\to\infty$. Note that some of the ε_{ik} may vanish; so, from Lemma 5.2, part 3), $\mathscr{G}(H_{fk})$ has characteristic polynomial of the form

$$\tau_k(s) = s^q+\varepsilon_{1k}s^{q-1}+\dots+\varepsilon_{qk},$$

where q may depend on k. Therefore, the limit of any convergent subsequence of $\{\tau_k\}$ is of the form s^q. Suppose $\hat{\tau}$ is the least common multiple of these limits; then $\hat{\tau}=s^{\hat{q}}$, where

$$\hat{q} = \overline{\lim_k}\, \deg(\tau_k) = \overline{\lim_k}\, \mu(\mathscr{G}(H_{fk})) = \overline{\lim_k}\, \mu(H_{fk}).$$

The last equality is obtained from Lemma 5.2, part 2). Arguing as in Lemma 5.2, part 2), $\delta\{\mathscr{G}(H_k)\}=\delta\{H_k\}$ for any $\{H_k\}$. Hence, from Theorem 6.8, we have

$$\delta\{H_{fk}\} = \delta\{\mathscr{G}(H_{fk})\} = \deg\hat{\tau} = \hat{q}.$$

Theorem 1.2, part 2) and Theorem 6.8 show that $\mu(H_{fk})=\deg\Delta_{fk}+\mu((H_k)_f)$. From Theorem 6.9,

$$\begin{aligned}\delta\{H_k\} &= \deg\hat{\Delta} + \overline{\lim}\,(\deg\Delta_{fk}+\mu((H_k)_f))\\ &= \overline{\lim_k}\,(\deg\hat{\Delta}+\deg\Delta_{fk}+\mu((H_k)_f))\\ &= \max_i\,\overline{\lim_k}\,(\deg\hat{\Delta}+\deg\Delta_{fk}^{(i)}+\mu((H_k^{(i)})_f)).\end{aligned}$$

It remains to prove

$$\deg\hat{\Delta}_k^{(i)} = \deg\hat{\Delta}+\deg\Delta_{fk}^{(i)}. \qquad (16)$$

By the definition of $\hat{\Delta}_k^{(i)}$,

$$\deg\hat{\Delta}_k^{(i)} = \deg\hat{\Delta}+\deg\Delta_k^{(i)}-\deg\Delta^{(i)}. \qquad (17)$$

Since $\deg\Delta_{sk}^{(i)}=\deg\Delta^{(i)}$, it follows from (15) that

$$\deg\Delta_k^{(i)} = \deg\Delta^{(i)}+\deg\Delta_{fk}^{(i)}. \qquad (18)$$

Combining (17) and (18), we obtain (16). □

Corollary 6.11 Suppose $H_k \to H$. If the CP of $\{H_k\}$ is convergent, then

$$\delta\{H_k\} = \overline{\lim_k}\, \mu(H_k).$$

Proof In this case, $\pi=1$ and $\hat{\Delta}_k^{(1)}=\Delta_k$, so

$$\begin{aligned}\delta\{H_k\} &= \overline{\lim_k}\, (\deg \Delta_k + \mu((H_k)_f)) \\ &= \overline{\lim_k}\, (\mu(H_k)_s + \mu((H_k)_f)) \\ &= \overline{\lim_k}\, \mu(H_k).\end{aligned}$$

Our next goal is to generalize part 3) of Theorem 1.2. An obvious conjecture is that a realization $\{\sigma_k\}$ of $\{H_k\}$ is minimal if and only if each σ_k is controllable and observable (as defined in [8]). While controllability and observability for every k (or, indeed, for infinitely many k) are clearly sufficient for minimality, the next examples demonstrate how necessity can fail.

Example 6.12 1) Even for a slow $\{H_k\}$, minimality of $\{\sigma_k\}$ does not imply controllability and observability even at a single point. Consider the sequence $\{H_k\}$ in Example 4.3, part 1). We can decompose $\{H_k\}$ into two subsequences

$$H_k^{(1)} = \frac{s+2}{(s+1)(s+2+\frac{1}{2k-1})}$$

$$H_k^{(2)} = \frac{s+3}{(s+1)(s+3+\frac{1}{2k})}.$$

For $H_k^{(1)}$ and $H_k^{(2)}$, the CP's are

$$\Delta_k^{(1)} = (s+1)(s+2+\frac{1}{2k-1}) \longrightarrow (s+1)(s+2) = \Delta^{(1)}$$

$$\Delta_k^{(2)} = (s+1)(s+3+\frac{1}{2k}) \longrightarrow (s+1)(s+3) = \Delta^{(2)}.$$

Thus

$$\hat{\Delta}(s) = \mathrm{LCM}\,\{\Delta^{(1)},\Delta^{(2)}\} = (s+1)(s+2)(s+3).$$

Since $\{H_k\}$ is a slow sequence, Theorem 6.9 indicates that $\delta\{H_k\}=\deg \hat{\Delta}=3$.

Every minimal realization $\{\sigma_k\}$ of $\{H_k\}$ must have dimension 3, but $\mu(\sigma_k)=2$ for each k; hence, no σ_k is controllable and observable.

2) In this example, the CP converges, but controllability and observability on a subsequence is the most that can be achieved. Let

$$H_k(s) = \begin{cases} 1, & k \text{ even} \\ \dfrac{1}{(\frac{1}{k}s+1)^2}, & k \text{ odd} \end{cases}$$

A simple calculation yields $\delta\{H_k\}=2$, but $\mu(H_k)=1$ when k is even, so any realization must contain infinitely many terms which are not controllable and observable.

The next result offers a weak extension of Theorem 1.2, part 3) to the sequential case.

<u>Theorem 6.13</u> Consider a convergent sequence $\{\sigma_k\}=\{(E_k,A_k,B_k,C_k)\}$, and let $H_k=C_k(sE_k-A_k)^{-1}B_k$. If the CP of $\{H_k\}$ converges and $\{\sigma_k\}$ is minimal, then there exists a subsequence $\{\sigma_{k_j}\}$ such that σ_{k_j} is controllable and observable for every j. If, in addition, $\{H_k\}$ is a slow sequence, then $\lim \sigma_k$ is controllable and observable, so σ_k is controllable and observable for every sufficiently large k.

<u>Proof</u> From Corollary 6.11, there exists a subsequence $\{H_{k_j}\}$ of $\{H_k\}$ such that $\delta\{H_k\}=\mu(H_{k_j})$ for all j. Therefore, each σ_{k_j} has dimension $\mu(H_{k_j})$ and must be controllable and observable. If $\{H_k\}$ is a slow sequence and the CP converges, Lemma 4.5, part 1) shows that $\Delta_k \rightarrow \Delta$, where Δ is the characteristic polynomial of $H=\lim H_k$. Since each Δ_k is monic, for large k we have $\mu(H_k)=\deg \Delta_k=\deg \Delta=\mu(H)$. From Corollary 6.11, $\dim \sigma=\delta\{H_k\}=\mu(H_k)$; hence, σ is controllable and observable. □

Restricting attention to slow sequences, Example 6.12, part 1) has special significance from an abstract algebraic perspective. It is easy to

show that algebraic controllability and observability over the ring c of convergent real sequences, as defined in [17, Chapter 2], is equivalent to controllability and observability of σ_k for sufficiently large k. Thus, linear systems over c do not satisfy the property that minimality implies algebraic controllability and observability.

We conclude this section by examining the problem of extending Theorem 1.2, part 4) to the sequential case. The following examples show that, in our case, two minimal realizations of the same sequence $\{H_k\}$ may not be related by nonsingular transformations (cf. [17, Theorem 4.19]).

Example 6.14 1) In fact, two minimal realizations may not be related by nonsingular transformation for any value of k. To see this, let $\{H_k\}$ again be the slow sequence given in Example 4.3, part 1). In Example 6.12, part 1), we showed that $\delta(H_k)=3$. Consider the two minimal realizations $\{(I, A_{1k}, B_k, C_{1k})\}$ and $\{(I, A_{2k}, B_k, C_{2k})\}$, where

$$A_{1k} = \begin{cases} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(6+\frac{3}{k}) & -(11+\frac{4}{k}) & -(6+\frac{1}{k}) \end{bmatrix}, & k \text{ even} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(6+\frac{2}{k}) & -(11+\frac{3}{k}) & -(6+\frac{1}{k}) \end{bmatrix}, & k \text{ odd} \end{cases}$$

$$A_{2k} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(6+\frac{5}{k}+\frac{1}{k^2}) & -(11+\frac{7}{k}+\frac{1}{k^2}) & -(6+\frac{2}{k}) \end{bmatrix}$$

$$B_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_{1k} = [6 \quad 5 \quad 1], \quad C_{2k} = \begin{cases} [6+\frac{2}{k} \quad 5+\frac{1}{k} \quad 1], & k \text{ even} \\ [6+\frac{3}{k} \quad 5+\frac{1}{k} \quad 1], & k \text{ odd} \end{cases}.$$

Suppose there exist convergent nonsingular matrix sequences $\{M_k\}$ and $\{N_k\}$ such that

$$M_kN_k = I,\ M_kA_{1k}N_{1k} = A_{2k}$$

$$M_kB_{1k} = B_{2k},\ C_{1k}N_k = C_{2k}$$

for each k. Then $N_k=M_k^{-1}$ and $A_{2k}=M_kA_{1k}M_k^{-1}$. But a simple calculation shows that A_{1k} and A_{2k} have different spectra, yielding a contradiction for each value of k.

2) When the CP converges, Theorem 6.13 implies that there exists a subsequence on which every minimal realization is controllable and observable. Hence, Theorem 1.2, part 4) guarantees that for any two minimal realizations there exist nonsingular sequences $\{M_k\}$ and $\{N_k\}$ which relate the various matrices. However, it may not be possible to find $\{M_k\}$ and $\{N_k\}$ which converge. Consider the sequence given by

$$H_k(s) = \frac{\frac{1}{k}}{\frac{1}{k}s+1}$$

with realizations $(\frac{1}{k},-1,\frac{1}{k},1)$ and $(\frac{1}{k},-1,1,\frac{1}{k})$. We have immediately $M_k\cdot\frac{1}{k}=1$, so $M_k=k$.

However, there is still one interesting case where a result is possible.

<u>Theorem 6.15</u> Suppose $\{H_k\}$ is a convergent sequence in $\mathbb{R}(s)^{rm}$ and let $\{(E_{ik},A_{ik},B_{ik},C_{ik})\}$: i=1,2 be two minimal realizations of $\{H_k\}$ with $(E_{ik},A_{ik},B_{ik},C_{ik})$ controllable and observable for every i,k. Further assume that each $(E_i,A_i,B_i,C_i)=\lim(E_{ik},A_{ik},B_{ik},C_{ik})$ is controllable and observable. Then there exist nonsingular matrix sequences $\{M_k\}$ and $\{N_k\}$ with $M_k \rightarrow M$ and $N_k \rightarrow N$, M and N nonsingular, such that $M_kE_{1k}N_k=E_{2k}$, $M_kA_{1k}N_k=A_{2k}$, $M_kB_{1k}=B_{2k}$, and $C_{1k}N_k=C_{2k}$ for every k.

<u>Proof</u> Applying the decomposition (7),(8) to $\{(E_{ik},A_{ik},B_{ik},C_{ik})\}$ yields decomposing matrices $\tilde{M}_{ik} \rightarrow \tilde{M}_i$ and $\tilde{N}_{ik} \rightarrow \tilde{N}_i$ and decomposed system matrices A_{isk}, A_{ifk}, B_{isk}, B_{ifk}, C_{isk}, and C_{ifk}, with $\lim A_{ifk}$ nilpotent. This determines in two ways the same time-scale decomposition $H_k=H_{sk}+H_{fk}$ given by

$$H_{sk}(s) = C_{isk}(sI-A_{isk})^{-1}B_{isk}$$
$$H_{fk}(s) = C_{ifk}(sA_{ifk}-I)^{-1}B_{ifk}.$$

Note that, for sufficiently large k, each of the subsystems $(I,A_{isk},B_{isk},C_{isk})$ and $(A_{ifk},I,B_{ifk},C_{ifk})$ must be controllable and observable. From [16,p. 208], the similarity transformation $T_{sk}=(V_{2sk}^TV_{2sk})^{-1}V_{2sk}^TV_{1sk}$, where V_{isk} is the observability matrix of the pair (A_{isk},C_{isk}), takes $(I,A_{1sk},B_{1sk},C_{1sk})$ into $(I,A_{2sk},B_{2sk},C_{2sk})$. Furthermore, $\{T_{sk}\}$ converges to the nonsingular matrix $T_s=(V_{2s}^TV_{2s})^{-1}V_{2s}^TV_{1s}$. A similar construction yields $T_{fk}\rightarrow T_f$. A straightforward calculation shows that the sequences

$$M_k = \tilde{M}_{2k}^{-1}\begin{bmatrix} T_{sk} & 0 \\ 0 & T_{fk} \end{bmatrix}\tilde{M}_{1k}, \quad N_k = \tilde{N}_{1k}\begin{bmatrix} T_{sk}^{-1} & 0 \\ 0 & T_{fk}^{-1} \end{bmatrix}\tilde{N}_{2k}^{-1}$$

yield the desired result.]

Our final result follows with the aid of Theorem 6.13.

Corollary 6.16 If $\{H_k\}$ is a slow sequence with convergent CP and $\{(E_{ik},A_{ik},B_{ik},C_{ik})\}$; i=1,2 are any two minimal realizations, then, for sufficiently large k, there exist nonsingular matrix sequences $\{M_k\}$ and $\{N_k\}$ and nonsingular matrices M and N such that $M_k\rightarrow M$, $N_k\rightarrow N$, $M_kE_{1k}N_k=E_{2k}$, $M_kA_{1k}N_k=A_{2k}$, $M_kB_{1k}=B_{2k}$, and $C_{1k}N_k=C_{2k}$ for every k.

7. Concluding Remarks

The problem discussed in this paper is the realization of convergent transfer matrix sequences with convergent generalized state-space sequences. Just as state-space sequences may be decomposed according to time-scale behavior, a time-scale decomposition for any rational matrix sequence may also be achieved. We have shown that convergence of the CP of a sequence of rational matrices is a crucial issue in the minimal realization problem. It was proved that, when the characteristic polynomial of a rational matrix

sequence is not convergent, the rational sequence can be decomposed into finitely many subsequences in such a way that each subsequence has convergent CP. Our results demonstrate that the general problem can be reduced to finitely many subproblems, each of which can be handled using a simpler theory. It is hoped that our results will complement the robustness literature at large.

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Technical Notes and Correspondence

Linear Compensator Designs Based Exclusively on Input-Output Information are Never Robust with Respect to Unmodeled Dynamics

J. DANIEL COBB

Abstract—**We investigate the effects of unmodeled, higher order dynamics or parasitics on the stability of linear control systems. We first describe a class of perturbations of a given state equation which cannot be distinguished from the original on the basis of input-output measurements alone. Then it is shown that, given any plant-compensator pair, such perturbations of each system can always be found which destabilize the closed-loop configuration. Finally, the effect of destabilizing perturbations on output behavior is explored.**

I. INTRODUCTION

The effects of high-frequency or parasitic phenomena on closed-loop system performance have long been studied. A popular framework for addressing this issue has been that of singular perturbation theory (see, e.g., [1], [2]). The point of view that parasitics are ultimately connected with unmodeled plant dynamics has become quite popular in recent years, sometimes with surprising consequences. For example, it was shown by Rohrs *et al.* [8] and Ioannou and Kokotovic [3] that high-frequency phenomena can lead to instability in adaptive control schemes. Adaptive controllers being highly nonlinear, a natural question to ask is whether parasitics could have a similar destabilizing effect on control systems which are based on linear compensators. This was answered in the affirmative by Khalil in [4] and [5]. A notable effort to circumvent these difficulties in the case of linear, time-invariant systems was made by Vidyasagar, culminating in the results of [6] and [7].

Our work is most similar to [7], but differs primarily in that we investigate the stability of a closed-loop system when *both* the plant and compensator are perturbed. The idea of perturbing both systems has been largely neglected in the literature (with the notable exception of [6]), even though one can easily make a strong case for considering such perturbations. Indeed, one need only recognize that a compensator, like the plant, is a physical system governed by a mathematical model which is inherently subject to uncertainty.

In light of examples such as those contained in [4] and [5], even arbitrarily small model errors are to be feared since such effects have the capability of destabilizing a system just as certainly as larger errors do. In fact, those examples illustrate that in some cases, small errors can cause greater instability than do larger ones.

In this paper, we intend to show that, when uncertainties in both plant and compensator are taken into account, even strictly proper compensators are subject to parasitic destabilization. Hence, properness of the compensator is really not the pivotal issue here as it is in [7]. We will show that, if only input-output information concerning the plant and compensator is available, robust compensation can never be achieved.

The results of this paper are by nature primarily negative. We do not claim to have a clear understanding yet of exactly what constitutes sufficient information for robust compensation, although we do mention a possible approach to finding an answer in Section V. It is hoped that our results will stimulate further discussion in an area which has been neglected by all but a handful of researchers.

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The author is with the Department of Electrical and Computer Engineering, University of Wisconsin, Madison, WI 53706.

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II. PRELIMINARIES

We study systems characterized by the linear, time-invariant state equations

$$\dot{x} = Ax + Bu, \; y = Cx + Du \tag{1}$$

and perturbations of (1) given by

$$\begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \; y = C_1 x + C_2 \xi \tag{2}$$

where the submatrices in (2) satisfy

$$A_{11} - A_{12} A_{22}^{-1} A_{21} = A, \; B_1 - A_{12} A_{22}^{-1} B_2 = B \tag{3}$$

$$C_1 - C_2 A_{22}^{-1} A_{21} = C, \; -C_2 A_{22}^{-1} B_2 = D \tag{4}$$

and A_{22} is nonsingular. If we set $\epsilon = 0$ in (2) and eliminate ξ, (1) is obtained; hence, (2) with $\epsilon = 0$ may be thought of as a state augmentation of (1). Setting $\epsilon > 0$ in (2) constitutes a perturbation of that augmentation. For the moment, we allow A_{22} to be either stable or unstable.

To aid our analysis, we will use the decomposition for singularly perturbed systems developed in [10] where it is shown that there exist real matrix-valued analytic maps $\epsilon \mapsto M_\epsilon$ and $\epsilon \mapsto N_\epsilon$, defined on some interval $[0, \beta)$, such that M_ϵ and N_ϵ are square and nonsingular for every ϵ and

$$M_\epsilon \begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix} N_\epsilon = \begin{bmatrix} I & 0 \\ 0 & A_{f\epsilon} \end{bmatrix}, \; M_\epsilon \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} N_\epsilon = \begin{bmatrix} A_{s\epsilon} & 0 \\ 0 & I \end{bmatrix} \tag{5}$$

with $A_{f\epsilon}$ and $A_{s\epsilon}$ analytic and A_{f0} nilpotent. According to [10], the matrices M_ϵ and N_ϵ are unique up to change of bases; hence, we may take M_0 and N_0 to be any matrices which achieve the decomposition (5) at $\epsilon = 0$. For example, let

$$M_0 = \begin{bmatrix} I & -A_{12} A_{22}^{-1} \\ 0 & I \end{bmatrix}, \; N_0 = \begin{bmatrix} I & 0 \\ -A_{22}^{-1} A_{21} & A_{22}^{-1} \end{bmatrix}.$$

Next, define

$$\begin{bmatrix} B_{s\epsilon} \\ B_{f\epsilon} \end{bmatrix} = M_\epsilon \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; [C_{s\epsilon} \;\; C_{f\epsilon}] = [C_1 \;\; C_2] N_\epsilon. \tag{6}$$

Equations (5) and (6) yield the decoupled state equations

$$\begin{bmatrix} I & 0 \\ 0 & A_{f\epsilon} \end{bmatrix} \begin{bmatrix} \dot{x}_s \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} A_{s\epsilon} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} + \begin{bmatrix} B_{s\epsilon} \\ B_{f\epsilon} \end{bmatrix} u$$

$$y = C_{s\epsilon} x_s + C_{f\epsilon} x_f \tag{7}$$

where

$$\begin{bmatrix} x_s \\ x_f \end{bmatrix} = N_\epsilon^{-1} \begin{bmatrix} x \\ \xi \end{bmatrix}.$$

We now present a series of technical results which will be useful in Sections III and IV.

Lemma 1: $A_{s0} = A$, $B_{s0} = B$, $C_{s0} = C$, $B_{f0} = B_2$, $C_{f0} = C_2A_{22}^{-1}$, and $A_{f\epsilon} = \epsilon F_\epsilon$ for every $\epsilon \in [0, \beta)$ where $F_0 = A_{22}^{-1}$.

Proof: From (5) and (6), we have

$$\begin{bmatrix} A_{s0} & 0 \\ 0 & I \end{bmatrix} = M_0 \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} N_0 = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix},$$

$$\begin{bmatrix} B_{s0} \\ B_{f0} \end{bmatrix} = M_0 \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B \\ B_2 \end{bmatrix}$$

$$[C_{s0} \quad C_{f0}] = [C_1 \quad C_2] N_0 = [C \quad C_2A_{22}^{-1}].$$

Let

$$M_\epsilon = \begin{bmatrix} M_{11\epsilon} & M_{12\epsilon} \\ M_{21\epsilon} & M_{22\epsilon} \end{bmatrix}, \; N_\epsilon^{-1} = \begin{bmatrix} \tilde{N}_{11\epsilon} & \tilde{N}_{12\epsilon} \\ \tilde{N}_{21\epsilon} & \tilde{N}_{22\epsilon} \end{bmatrix}$$

and note that

$$N_0^{-1} = \begin{bmatrix} I & 0 \\ A_{21} & A_{22} \end{bmatrix}, \; M_\epsilon \begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A_{f\epsilon} \end{bmatrix} N_\epsilon^{-1}.$$

We thus have $\epsilon M_{22\epsilon} = A_{f\epsilon}\tilde{N}_{22\epsilon}$ so $A_{f\epsilon} = \epsilon F_\epsilon$ where $F_0 = M_{220}\tilde{N}_{220}^{-1} = A_{22}^{-1}$. □

From $A_{f\epsilon} = \epsilon F_\epsilon$, we immediately obtain the well-known result that the eigenvalues of (2) which tend to infinity as $\epsilon \to 0^+$ are "close" to those of $(1/\epsilon)A_{22}$ (see, e.g., [2, Corollary 2.1]). One useful way of stating this result is the following.

Lemma 2: If μ is an eigenvalue of A_{22}, $\gamma > 0$, and $R < \infty$, then there exists $\epsilon_0 > 0$ such that (2) has an eigenvalue λ_ϵ satisfying $|\lambda_\epsilon| > R$ and $|\arg \lambda_\epsilon - \arg (1/\epsilon)\mu| < \gamma$ whenever $0 < \epsilon < \epsilon_0$.

Proof: From (7), the eigenvalues of $(1/\epsilon)F_\epsilon^{-1}$ are also eigenvalues of (2). Since $F_0^{-1} = A_{22}$ and F_ϵ^{-1} is continuous in ϵ, each F_ϵ^{-1} has an eigenvalue μ_ϵ with $\mu_\epsilon \to \mu$ as $\epsilon \to 0^+$. Choose ϵ_0 so that $(1/\epsilon)|\mu_\epsilon| > R$ and $|\arg \mu_\epsilon - \arg \mu| < \gamma$ whenever $0 < \epsilon < \epsilon_0$, and let $\lambda_\epsilon = (1/\epsilon)\mu_\epsilon$. Then λ_ϵ is an eigenvalue of (2), $|\lambda_\epsilon| > R$, and $|\arg \lambda_\epsilon - \arg (1/\epsilon)\mu| = |\arg \mu_\epsilon - \arg \mu| < \gamma$. □

Suppose the transfer matrices of (1) and (2) are P and P_ϵ, respectively. We will need conditions under which an eigenvalue of (2) is also a pole of P_ϵ.

Lemma 3: If (A_{22}, B_2, C_2) is controllable and observable, there exists $\epsilon_0 > 0$ and $R < \infty$ such that every eigenvalue λ_ϵ of (2) satisfying $|\lambda_\epsilon| > R$ is also a pole of P_ϵ whenever $0 < \epsilon < \epsilon_0$.

Proof: An eigenvalue λ_ϵ of (2) is a pole of P_ϵ if

$$\begin{bmatrix} \lambda_\epsilon I - A_{11} & -A_{12} & B_1 \\ -A_{21} & \epsilon\lambda_\epsilon I - A_{22} & B_2 \end{bmatrix} = M_\epsilon^{-1} \begin{bmatrix} \lambda_\epsilon I - A_{s\epsilon} & 0 & B_{s\epsilon} \\ 0 & \epsilon\lambda_\epsilon I - F_\epsilon^{-1} & B_{f\epsilon} \end{bmatrix} \cdot \begin{bmatrix} \begin{bmatrix} I & 0 \\ 0 & F_\epsilon \end{bmatrix} N_\epsilon^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{8}$$

and

$$\begin{bmatrix} \lambda_\epsilon I - A_{11} & -A_{12} \\ -A_{21} & \lambda_\epsilon I - A_{22} \\ C_1 & C_2 \end{bmatrix} = \begin{bmatrix} M_\epsilon^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda_\epsilon I - A_{s\epsilon} & 0 \\ 0 & \epsilon\lambda_\epsilon I - F_\epsilon^{-1} \\ C_{s\epsilon} & C_{f\epsilon}F_\epsilon^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & F_\epsilon \end{bmatrix} N_\epsilon^{-1} \tag{9}$$

have full rank. Choose $R > \max \{|\lambda| \mid \lambda \text{ is an eigenvalue of } A\}$. From Lemma 1, $(F_0^{-1}, B_{f0}, C_{f0}F_0^{-1}) = (A_{22}, B_2, C_2)$. Hence, there exists $\epsilon_0 > 0$ such that, whenever $0 < \epsilon < \epsilon_0$, $(F_\epsilon^{-1}, B_{f\epsilon}, C_{f\epsilon}F_\epsilon^{-1})$ is controllable and observable and $|\lambda_\epsilon| > R$ implies that λ_ϵ is not an eigenvalue of $A_{s\epsilon}$. It follows immediately that all matrices on the right-hand sides of (8) and (9) have full rank. □

III. Input-Output Equivalence

In this section, we explore the relationship between the nominal and perturbed systems (1) and (2) and discuss the conditions under which they are indistinguishable if only input-output information is available. Consider the process of obtaining or verifying an input-output model of a physical system. We are allowed to take measurements by applying an input signal starting at $t = 0$ and by observing the output; it is assumed that no direct access to internal states is possible. Once a nominal model is obtained, a controllable and observable realization can be chosen, yielding the state equation (1). Since we have no direct control over initial states except through the input ports, and since $t = 0$ presumably occurs long after the system was built, the system may be assumed initially at rest. Hence, we choose $x(0) = 0$ and $\xi(0) = 0$ in (1) and (2).

We define the class of admissible input signals $\mathfrak{U}$ to be all C^1 functions $u:[0, \tau] \to \Re^m$ satisfying $\max \|u(t)\| < K_0$, $\max \|\dot{u}(t)\| < K_1$, and $u(0) = 0$ where the constants $\tau < \infty$, $K_0 < \infty$, and $K_1 < \infty$ are independent of u. From an engineering standpoint, it is not unreasonable to place such restrictions on u. Indeed, in any real-world scenario, there is a maximum length of time one would be willing to invest in collecting data, as well as a maximum amplitude of voltage, force, or other input quantity that could possibly be generated using available technology. Furthermore, there is always an upper bound on the rate at which $u(t)$ can be made to vary (e.g., every amplifier has a maximum slew rate). Thus, the constants τ, K_0, and K_1, although possibly very large, must be finite. Since no input is applied prior to $t = 0$ and since $K_1 < \infty$, we must have $u(0) = 0$. We would surely be in serious trouble if, in order to design a robust compensator, we needed the capability of generating inputs over arbitrarily large intervals of time or with arbitrarily large amplitudes or rates of change.

Associated with any real-world measuring device is a minimum error which can be detected. For example, if a function y represents an output voltage, velocity, or other physical quantity of interest, there must exist a number $\delta > 0$, characteristic of the measuring device alone, such that another output $\tilde{y}$ cannot be distinguished from y if

$$\sup \{\|y(t) - \tilde{y}(t)\| \quad 0 \le t \le \tau\} < \delta. \tag{10}$$

For the remainder of the paper, we assume a fixed source of input signals and measurements and, consequently, a fixed set $\mathfrak{U}$ and number $\delta > 0$.

The quantities $\mathfrak{U}$ and δ together determine an equivalence between systems: two systems are indistinguishable under input-output measurement if for every $u \in \mathfrak{U}$, the output functions y and $\tilde{y}$ of the two systems satisfy (10). The next result applies this idea to the nominal and perturbed models (1) and (2).

Theorem 1: If A_{22} is strictly stable, there exists $\epsilon_0 > 0$ such that, whenever $u \in \mathfrak{U}$ and $0 \le \epsilon < \epsilon_0$, the respective outputs y and y_ϵ of (1) and (2) satisfy $\max \{\|y(t) - y_\epsilon(t)\| \mid 0 \le t \le \tau\} < \delta$.

Proof: We first note that $y_0(t) = \int_0^t C_{s0} \exp(\eta A_{s0})B_{s0}u(t - \eta)\, d\eta - C_{f0}B_{f0}u(t) = y(t)$. Hence, we need only show that there exists ϵ_0 such that $\|y_\epsilon(t) - y_0(t)\| < \delta$ whenever $0 \le t \le \tau$ and $0 \le \epsilon < \epsilon_0$. Decomposing $y_\epsilon = y_{s\epsilon} + y_{f\epsilon}$ in the obvious way, we have $\|y_{s\epsilon}(t) - y_{s0}(t)\| \le K_0 \int_0^\tau \|C_{s\epsilon} \exp(\eta A_{s\epsilon})B_{s\epsilon} - C_{s0} \exp(\eta A_{s0})B_{s0}\|\, d\eta$. Choose $\epsilon_1 > 0$ such that $0 \le \epsilon < \epsilon_1$ implies $\max \{\|C_{s\epsilon} \exp(\eta A_{s\epsilon})B_{s\epsilon} - C_{s0} \exp(\eta A_{s0})B_{s0}\| \mid 0 \le \eta \le \tau\} < \delta/(2K_0\tau)$. Integrating by parts, we obtain

$$\|y_{f\epsilon}(t) - y_{f0}(t)\| \le K_1 \|C_{f\epsilon}\| \int_0^\tau \left\| \exp\left(\frac{\eta}{\epsilon} F_\epsilon^{-1}\right) \right\| d\eta \cdot \|B_{f\epsilon}\| + K_0 \|C_{f\epsilon}B_{f\epsilon} - C_{f0}B_{f0}\|.$$

There exist $\epsilon_2 > 0$ and $K < \infty$ such that $\|\exp(tF_\epsilon^{-1})\| < K$, $\|C_{f\epsilon}\| < K$, and $\|B_{f\epsilon}\| < K$ whenever $t \ge 0$ and $0 \le \epsilon < \epsilon_2$. Let $\bar{\delta} = \delta/(4K_1K^2(K + \tau))$. We know that there exists $\epsilon_3 > 0$ such that $\|\exp((\eta/\epsilon)F_\epsilon^{-1})\| < \bar{\delta}$ whenever $\bar{\delta} \le \eta \le \tau$ and $0 < \epsilon < \epsilon_3$ (see, e.g., [13]). Finally, there exists $\epsilon_4 > 0$ such that $\|C_{f\epsilon}B_{f\epsilon} - C_{f0}B_{f0}\| < \delta/4K_0$ when $\epsilon < \epsilon_4$. Let $\epsilon_0 = \min \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$. Then $0 \le \epsilon < \epsilon_0$ implies $\|y_\epsilon(t) - y_0(t)\| < \delta/2 + K_1K^2(K\bar{\delta} + \tau\bar{\delta}) + \delta/4 = \delta$. □

We have thus established that, for sufficiently small ϵ, (1) and (2) are indistinguishable on the basis of input-output information. Hence, although the physical system is nominally described by (1), an equally

valid model from an input-output perspective is given by (2) with ϵ sufficiently small and A_{22} strictly stable.

IV. Closed-Loop Destabilization

We are now ready to investigate the effects that the system perturbations in Section II have on a closed-loop configuration. Consider the feedback compensator governed by

$$\dot{z}=Fz+Gy,\ u=Hz+v. \tag{11}$$

We consider only compensators with strictly proper transfer matrices since the results of [7] indicate that nonstrictly proper compensators are never robust with respect to unmodeled dynamics. Perturbations of (11) are of the form

$$\begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix}\begin{bmatrix} \dot{z} \\ \dot{\zeta} \end{bmatrix}=\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\begin{bmatrix} z \\ \zeta \end{bmatrix}+\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} y$$
$$u=H_1z+H_2\zeta+v \tag{12}$$

where

$$F_{11}-F_{12}F_{22}^{-1}F_{21}=F,\ G_1-F_{12}F_{22}^{-1}G_2=G \tag{13}$$

$$H_1-H_2F_{22}^{-1}F_{21}=H,\ -H_2F_{22}^{-1}G_2=0 \tag{14}$$

and F_{22} is nonsingular. The discussion of Section III applies equally well to both plant and compensator.

Combining (1) and (13) in a standard feedback configuration yields

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix}=\begin{bmatrix} A & BH \\ GC & F+GDH \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}+\begin{bmatrix} B \\ GD \end{bmatrix} v$$
$$y=Cx+DHz. \tag{15}$$

Combining the perturbed systems (2) and (12) gives

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \epsilon I & 0 \\ 0 & 0 & 0 & \epsilon I \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{\xi} \\ \dot{\zeta} \end{bmatrix}$$
$$=\begin{bmatrix} A_{11} & B_1H_1 & A_{12} & B_1H_2 \\ G_1C_1 & F_{11} & G_1C_2 & F_{12} \\ A_{21} & B_2H_1 & A_{22} & B_2H_2 \\ G_2C_1 & F_{21} & G_2C_2 & F_{22} \end{bmatrix}\begin{bmatrix} x \\ z \\ \xi \\ \zeta \end{bmatrix}+\begin{bmatrix} B_1 \\ 0 \\ B_2 \\ 0 \end{bmatrix} v$$
$$y=C_1x+C_2z. \tag{16}$$

Let (15) and (16) have transfer matrices H and H_ϵ, respectively.

From this point on, we assume that A_{22} and F_{22} are strictly stable matrices. Thus, according to Theorem 1, (2) and (12) are equivalent to (1) and (11) for sufficiently small ϵ in an input-output sense. The perturbed closed-loop system (16) is also of the form (2); no obvious conclusions can be drawn, however, concerning stability of either (16) or the matrix

$$X=\begin{bmatrix} A_{22} & B_2H_2 \\ G_2C_2 & F_{22} \end{bmatrix}.$$

In view of Lemmas 2-4 as related to (16), we see that the properties of X as well as those of the matrices

$$Y=\begin{bmatrix} B_2 \\ 0 \end{bmatrix},\ Z=[C_2\ \ 0]$$

are crucial for understanding the behavior of (16).

We are ultimately interested not only in the eigenvalues of the closed-loop system, but also in the poles of H_ϵ and the behavior of the system output $y(t)$. The next two results treat first the closed-loop poles and then output behavior. As a means of quantifying instability, let $\alpha \in (0, \pi/2)$ and consider the open sector $S = \{s \in \mathbb{C} - \{0\} \mid |\arg s| < \alpha\}$.

Theorem 2: Suppose $R < \infty$, (X, Y, Z) is controllable and observable, and X is nonsingular with an eigenvalue in the sector S. Then there exists $\epsilon_0 > 0$ such that H_ϵ has a pole $p_\epsilon \in S$ satisfying $|p_\epsilon| > R$ whenever $0 < \epsilon < \epsilon_0$.

Proof: Since X is nonsingular, the closed-loop system (16) is of the form (2). Let $\mu \in S$ be an eigenvalue of X. There exists $\gamma > 0$ such that $s \in S$ whenever $|\arg s - \arg (1/\epsilon)\mu| < \gamma$. The result then follows from Lemmas 2 and 3. □

Now consider behavior of the output $y(t)$ in the closed-loop system (16). Theorem 3 shows that under certain conditions, the instability described in Theorem 2 also has a pronounced effect on $y(t)$. Let m denote Lebesgue measure.

Theorem 3: Suppose $R < \infty$, $\delta_1, \delta_2 > 0$, (X, Y, Z) is controllable and observable, and X is nonsingular with an eigenvalue in the sector S.

1) There exists $\epsilon_0 > 0$ such that corresponding to each $\epsilon \in (0, \epsilon_0)$, there exist vectors $x_{0\epsilon} \in \mathbb{R}^n$, $z_{0\epsilon} \in R^k$, $\xi_{0\epsilon} \in R^{\bar{n}}$, $\zeta_{0\epsilon} \in \mathbb{R}^{\bar{n}}$ with $\|x_{0\epsilon}\|$, $\|z_{0\epsilon}\|$, $\|\xi_{0\epsilon}\|$, $\|\zeta_{0\epsilon}\| < \delta_1$ and a set $\Omega_\epsilon \subset [0, \tau]$ with $m\Omega_\epsilon < \delta_2$ such that the output y_ϵ of (20), subject to $x(0) = x_{0\epsilon}$, $z(0) = z_{0\epsilon}$, $\xi(0) = \xi_{0\epsilon}$, $\zeta(0) = \zeta_{0\epsilon}$, and $u \equiv 0$, satisfies $\|y_\epsilon(t)\| > R$ for every $t \in [0, \tau] - \Omega_\epsilon$.

2) There exists $\epsilon_0 > 0$ such that corresponding to each $\epsilon \in (0, \epsilon_0)$, there exist a continuous function $u_\epsilon:[0, \tau] \to \mathbb{R}^m$ with $\|u_\epsilon(t)\| < \delta_1$ for all $t \in [0, \tau]$ and a set $\Omega_\epsilon \subset [0, \tau]$ with $m\Omega_\epsilon < \delta_2$ such that the output of (20), subject to $x(0) = z(0) = \xi(0) = \zeta(0) = 0$ and $u \equiv u_\epsilon$, satisfies $\|y_\epsilon(t)\| > R$ for every $t \in [0, \tau] - \Omega_\epsilon$.

Proof: 1) Since R is arbitrary and the system (16) is linear, we need only prove the result for a single vector norm, say, the Euclidean norm. The decomposition (7) may be applied to (16), yielding real-valued analytic matrix functions $M_\epsilon, N_\epsilon, A_{s\epsilon}, B_{s\epsilon}, \cdots, F_\epsilon$ defined on an interval $[0, \beta]$. Since $F_0 = X^{-1}$ is nonsingular, F_ϵ^{-1} is analytic. It is shown in [15] that there exists a continuous complex unitary matrix-valued function $\epsilon \to U_\epsilon$ defined for sufficiently small values of ϵ that puts F_ϵ^{-1} into continuous upper triangular form—i.e.,

$$U_\epsilon^{-1}F_\epsilon^{-1}U_\epsilon=\begin{bmatrix} \mu_{1\epsilon} & \alpha_{12\epsilon} & \cdots & \alpha_{1,\bar{n}+\bar{k},\epsilon} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \alpha_{\bar{n}+\bar{k}-1,\bar{n}+\bar{k},\epsilon} \\ 0 & \cdots & 0 & \mu_{\bar{n}+\bar{k},\epsilon} \end{bmatrix}$$

where each of the maps $\epsilon \to \mu_{i\epsilon}$ and $\epsilon \to \alpha_{ij\epsilon}$ is continuous. Additional row and column interchanges can be used to reindex the $\mu_{i\epsilon}$; equivalently, U_0 may be chosen so that $\mu_{10} \in S$.

Let

$$w_\epsilon=\frac{\delta_1}{2\|N_\epsilon\|}N_\epsilon\begin{bmatrix} 0 \\ U_\epsilon\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{bmatrix}.$$

Since N_ϵ is nonsingular on $[0, \beta)$, $\|N_\epsilon\|$ is nonzero. Standard norm inequalitites reveal that $\|W_\epsilon\| < \delta_1$. From (7), it follows that the natural response of (16) due to the initial condition w_ϵ is

$$\tilde{y}_\epsilon(t)=\frac{\delta_1}{2\|N_\epsilon\|}(C_{f\epsilon}U_\epsilon)\exp\left(\tfrac{t}{\epsilon}U_\epsilon^{-1}F_\epsilon^{-1}U_\epsilon\right)\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{17}$$

From Lemma 1, $(F_0^{-1}, C_{f0}) = (X, ZX^{-1})$. This pair is observable since X is nonsingular; the corresponding observability matrix is

$$\begin{bmatrix} ZX^{-1} \\ Z \\ ZX \\ \vdots \\ ZX^{\bar{n}+\bar{k}-2} \end{bmatrix}=\begin{bmatrix} Z \\ ZX \\ \vdots \\ ZX^{\bar{n}+\bar{k}-1} \end{bmatrix}X^{-1}$$

and the pair (X, Z) is observable. Thus, $(U_0^{-1}F_0^{-1}U_0,\ C_{f0}U_0)$ is observable. Since $U_0^{-1}F_0^{-1}U_0$ is upper triangular, the first column of $C_{f0}U_0$ is nonzero. Suppose $\alpha_0 \neq 0$ is the ith entry of the first column of $C_{f0}U_0$. Then the same entry α_ϵ of $C_{f\epsilon}U_\epsilon$ is nonzero for sufficiently small

ϵ. From (17), it follows that $\hat{y}_\epsilon$ has ith entry $\hat{y}_{i\epsilon}(t) = (\delta_1/(2\|N_\epsilon\|))\alpha_\epsilon \exp((1/\epsilon)\mu_{1\epsilon}t)$. Thus, Re $\hat{y}_\epsilon$ has the ith entry

$$\mathrm{Re}\,\hat{y}_{i\epsilon}(t) = (\delta_1/(2\|N_\epsilon\|))|\alpha_\epsilon| \exp\left(\frac{1}{\epsilon}\,\mathrm{Re}\,\mu_{1\epsilon}t\right)\cos\left(\frac{1}{\epsilon}\,\mathrm{Im}\,\mu_{1\epsilon}t + \arg\alpha_\epsilon\right).$$

Since $\mu_{10} \in S$, Re $\mu_{1\epsilon} > 0$ for small ϵ. From elementary analysis, there exists $\epsilon_0 > 0$ such that $0 < \epsilon < \epsilon_0$ implies the existence of a set Ω_ϵ with $m\Omega_\epsilon < \delta_2$ and $\|\mathrm{Re}\,\hat{y}_\epsilon(t)\| > R$ for all $t \in [0, \tau] - \Omega_\epsilon$.

We note that the initial condition w_ϵ may be complex. In general, the natural response of (16) is of the form $y(t) = \Gamma_\epsilon(t)w$ where $\Gamma_\epsilon(t)$ is a real-valued matrix. Hence, Re $y(t) = \Gamma_\epsilon(t)$ Re w, and if we set $[x_{0\epsilon} z_{0\epsilon} \xi_{0\epsilon} \zeta_{0\epsilon}]^T$ = Re w_ϵ^T, we obtain an output y_ϵ with ith entry $y_{i\epsilon}(t)$ = Re $\hat{y}_{i\epsilon}(t)$. Therefore, $\|y_\epsilon(t)\| \geq |\hat{y}_{i\epsilon}(t)| > R$ for all $t \in [0, \tau] - \Omega_\epsilon$. Finally, we note that $\|x_{0\epsilon}\|, \|z_{0\epsilon}\|, \|\xi_{0\epsilon}\|, \|\zeta_{0\epsilon}\| \leq \|\mathrm{Re}\,w_\epsilon\| \leq \|w_\epsilon\| < \delta_1$.

2) Our approach is to construct an input function u_ϵ which steers the system (20) from the origin to some state w_ϵ satisfying the conditions of part 1), the transfer occurring on an arbitrarily small t interval; then the system will be allowed to evolve from w_ϵ with zero input. We first consider the pair $(F_\epsilon^{-1}, F_\epsilon^{-1}B_{f\epsilon})$. From Lemma 1, $(F_0^{-1}, F_0^{-1}B_{f0}) = (X, XY)$. This pair is controllable since X is nonsingular; the corresponding controllability matrix is

$$[XY \;\; X^2Y \;\; \cdots \;\; X^{\hat{n}+\hat{k}}Y] = X[Y \;\; XY \;\; \cdots \;\; X^{\hat{n}+\hat{k}-1}Y]$$

and (X, Y) is controllable. Hence, $(F_\epsilon^{-1}, F_\epsilon^{-1}B_{f\epsilon})$ is controllable for sufficiently small ϵ. Let

$$\psi_\epsilon(t) = B_{f\epsilon}^T F_\epsilon^{-T} \exp(-tF_\epsilon^{-T}) W_\epsilon(\tau)^{-1} \exp(-\tau F_\epsilon^{-T}) \tag{18}$$

where the Gramian $W_\epsilon(\tau)$ is given by $W_\epsilon(\tau) = \int_0^\tau \exp(-\eta F_\epsilon^{-1}) F_\epsilon^{-1}B_{f\epsilon}B_{f\epsilon}^T F_\epsilon^{-T} \exp(-\eta F_\epsilon^{-T})\, d\eta$. W_ϵ is nonsingular for small ϵ since $(F_\epsilon^{-1}, F_\epsilon^{-1}B_{f\epsilon})$ is controllable (see [11, p. 184]). All matrices in (18) converge and $\exp(-tF_\epsilon^{-T})$ converges uniformly on $[0, \tau]$ as $\epsilon \to 0^+$; hence, ψ_ϵ converges uniformly to ψ_0. Thus, there exists a number $M_1 < \infty$ such that $\|\psi_\epsilon(t)\| < M_1$ for all $t \in [0, \tau]$ and ϵ sufficiently small.

Choose $M_2 < \infty$ such that $\|C_{s\epsilon} \exp(tA_{s\epsilon})\| < M_2$ for small ϵ and all $t \in [0, \tau]$ where $C_{s\epsilon}$ and $A_{s\epsilon}$ are given by (7). Since N_ϵ^{-1} is continuous, we know from part 1) that for sufficiently small ϵ, there exist real vectors $x_{0\epsilon}$, $z_{0\epsilon}$, $\xi_{0\epsilon}$, and $\zeta_{0\epsilon}$ with $\|x_{0\epsilon}\|, \|z_{0\epsilon}\|, \|\xi_{0\epsilon}\|, \|\zeta_{0\epsilon}\| < \delta_1/(2M_1\|N_\epsilon^{-1}\|)$ and a set $\hat{\Omega}_\epsilon$ with $m\hat{\Omega}_\epsilon < \delta_2/2$ such that the corresponding output $\hat{y}_\epsilon$ of (16) satisfies $\|\hat{y}_\epsilon(t)\| > R + (M_2/M_1)\delta_1$ for every $t \in [0, \tau] - \hat{\Omega}_\epsilon$. Let

$$\begin{bmatrix} x_{0s\epsilon} \\ x_{0f\epsilon} \end{bmatrix} = N_\epsilon^{-1} \begin{bmatrix} x_{0\epsilon} \\ z_{0\epsilon} \\ \xi_{0\epsilon} \\ \zeta_{0\epsilon} \end{bmatrix}. \tag{19}$$

Then the output $\hat{y}_\epsilon$ may be written $\hat{y}_\epsilon = y_{s\epsilon} + y_{f\epsilon}$ where $y_{s\epsilon}(t) = C_{s\epsilon} \exp(tA_{s\epsilon})x_{0s\epsilon}$ and $y_{f\epsilon}(t) = C_{f\epsilon} \exp((t/\epsilon)F_\epsilon^{-1})x_{0f\epsilon}$. From (19), $\|x_{0s\epsilon}\| < \delta_1/M_1$; therefore, $\|y_{s\epsilon}(t)\| < (M_2/M_1)\delta_1$ for every $t \in [0, \tau]$. It follows that $\|y_{f\epsilon}(t)\| > \|\hat{y}_\epsilon(t)\| - \|y_{s\epsilon}(t)\| > R$ for each $t \in [0, \tau] - \hat{\Omega}_\epsilon$.

Next, define $\hat{u}_\epsilon(t) = \psi_\epsilon(t)x_{0f\epsilon}$. Then $\|x_{0f\epsilon}\| < \delta_1/M_1$ guarantees that $\|\hat{u}_\epsilon(t)\| < \delta_1$, and $\hat{u}_\epsilon$ steers the system $\dot{x} = F_\epsilon^{-1}x + F_\epsilon^{-1}B_{f\epsilon}u$ from the origin at $t = 0$ to $x_{0f\epsilon}$ at $t = \tau$. (See, e.g., [11, p. 556].) Let

$$u_\epsilon(t) = \begin{cases} \hat{u}_\epsilon\left(\frac{t}{\epsilon}\right), & 0 \leq t \leq \epsilon\tau \\ 0, & \epsilon\tau < t \leq \tau. \end{cases}$$

Then $\|u_\epsilon(t)\| < \delta_1$ and u_ϵ steers the second subsystem in (7) from the origin at $t = 0$ to $x_{0f\epsilon}$ at $t = \epsilon\tau$. u_ϵ also steers the first subsystem in (7) from the origin to some state $\tilde{x}_{0s\epsilon}$ at $t = \epsilon\tau$. Since $\tilde{x}_{0s\epsilon}$ is given by the convolution integral $\tilde{x}_{0s\epsilon} = \int_0^{\epsilon\tau} \exp(tA_{s\epsilon})B_{s\epsilon}u_\epsilon(t)\,dt$, the construction of u_ϵ and uniform convergence of $\exp(tA_{s\epsilon})$ guarantee that $\tilde{x}_{0s\epsilon} \to 0$ as $\epsilon \to 0^+$. Hence, $\tilde{y}_{s\epsilon} \to 0$ uniformly on $[0, \tau]$ as $\epsilon \to 0^+$ where $\tilde{y}_{s\epsilon}(t) = C_{s\epsilon} \exp(tA_{s\epsilon})\tilde{x}_{0s\epsilon}$. Applying the input u_ϵ steers the system (20) to $w_\epsilon = N_\epsilon[\tilde{x}_{0s\epsilon} x_{0f\epsilon}]^T$ at $t = \epsilon\tau$. For $t \in [\epsilon\tau, \tau]$, the corresponding output is $y_\epsilon(t) = \tilde{y}_{s\epsilon}(t - \epsilon\tau) + y_{f\epsilon}(t - \epsilon\tau)$, so $\|y_\epsilon(t)\| > R - \|\tilde{y}_{s\epsilon}(t - \epsilon\tau)\| > R$ for small ϵ and all $t \in [\epsilon\tau, \tau] - (\epsilon\tau + \hat{\Omega}_\epsilon)$. Thus, if we choose ϵ_0 sufficiently small with $\epsilon_0 < \delta_2/2$, and $\Omega_\epsilon = [0, \epsilon\tau] \cup (\epsilon\tau + \hat{\Omega}_\epsilon)$, we obtain $m\Omega_\epsilon < \delta_2$ and $\|y_\epsilon(t)\| > R$ for all $t \in [0, \tau] - \Omega_\epsilon$ whenever $0 < \epsilon < \epsilon_0$. □

The divergence of the output of the closed-loop system described in Theorem 3 is referred to in analysis texts as "almost uniform convergence to infinity." In view of the arbitrarily tight bounds that may be placed on an input or initial condition which generate this divergent behavior, we conclude that, if the assumptions of the theorem are met, unbounded instability at the output of a closed-loop configuration can result from arbitrarily small noise impinging on the system.

So far we have demonstrated that the existence of destabilizing perturbations of the plant and compensator is guaranteed if a certain linear algebra problem admits a solution. Indeed, if any A_{22}, B_2, and C_2 are chosen, (3) and (4) may be satisfied by simply selecting A_{12} and A_{21} arbitrarily and solving for A_{11}, B_1, and C_1. A similar remark applies to (13) and (14). It is sufficient, therefore, to find A_{22}, B_2, C_2, F_{22}, G_2, and H_2 such that 1) A_{22} and F_{22} are strictly stable, 2) (X, Y, Z) is controllable and observable, 3) X is nonsingular with an eigenvalue in S, and 4) (4) and (14) are satisfied. Theorems 2 and 3 further indicate that, if 1)–4) are met, the resulting instability in (16) becomes progressively worse as $\epsilon \to 0$ since R may be chosen arbitrarily large. *Thus, arbitrarily small uncertainty can lead to arbitrarily large instability.*

We now address the linear algebra problem 1)–4). We really need to find only one solution in order to demonstrate the existence of destabilizing perturbations; however, it is possible to do better. To obtain an understanding of just how many destabilizing perturbations actually exist, let (1), (2), (11), and (12) have orders n, $n + \hat{n}$, k, and $k + \hat{k}$, respectively; define $q = (n + \hat{n})(n + \hat{n} + m + p) + (k + \hat{k})(k + \hat{k} + p + m)$. Also, consider the variety in $\mathfrak{R}^q$ consisting of all $(A_{11}, \cdots, C_2, F_{11}, \cdots, H_2)$ such that (3)–(6) and (13) and (14) are satisfied, and let $V \subset \mathfrak{R}^q$ denote the intersection of that variety with the subset in which A_{22} and F_{22} are strictly stable. V may be interpreted as the set of all possible state augmentations of (1) and (11) of order $\hat{n}$ and $\hat{k}$, respectively. Finally, let $\Gamma \subset \mathfrak{R}^q$ be the set of all points for which (X, Y, Z) is controllable and observable and X is nonsingular with an eigenvalue in S. We are interested in properties of the set $V \cap \Gamma$.

Theorem 4:

1) $V \cap \Gamma$ is relatively open in V.

2) $V \subset \Gamma$ is nonempty if $\hat{k} \geq 2$ and either a) $D = 0$ and $\hat{n} \geq 2$ or b) $D \neq 0$ and $\hat{n} \geq$ rank D.

Proof:

1) This is obvious since Γ is open in $\mathfrak{R}^q$.

2) Suppose $D = 0$ and consider

$$T(s) = \begin{bmatrix} 2^{\hat{n}}s/(s+1)^{\hat{n}} & 0 \cdots 0 \\ 0 & \vdots \\ \vdots & \\ 0 \cdots\cdots & 0 \end{bmatrix}, \quad U(s) = \begin{bmatrix} 2^{\hat{k}}s/(s+1)^{\hat{k}} & 0 \cdots 0 \\ 0 & \vdots \\ \vdots & \\ 0 \cdots\cdots & 0 \end{bmatrix}.$$

Let (A_{22}, B_2, C_2) and (F_{22}, G_2, H_2) be controllable and observable realizations of T and U, respectively. Then A_{22} and F_{22} are strictly stable, $-C_2A_{22}^{-1}B_2 = T(0) = D$, and $-H_2F_{22}^{-1}G_2 = U(0) = 0$. Note that T and U have degrees $\hat{n}$ and $\hat{k}$. Since (X, Y, Z) has transfer function

$$V(s) = (I - T(s)U(s))^{-1}T(s)$$

$$= \begin{bmatrix} 2^{\hat{k}}s(s+1)^{\hat{n}}/((s+1)^{\hat{n}+\hat{k}} - 2^{\hat{n}+\hat{k}}s) & 0 \cdots 0 \\ 0 & \vdots \\ \vdots & \\ 0 \quad \cdots & 0 \end{bmatrix}$$

and V has characteristic polynomial $\Delta(s) = (s+1)^{\hat{n}+\hat{k}} - 2^{\hat{n}+\hat{k}}s$, it follows that (X, Y, Z) is controllable and observable and X is nonsingular with a unit eigenvalue.

Now suppose $D \neq 0$. There exist nonsingular matrices M and N such that

$$MDN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where r = rank D. Let

$$T(s)=M^{-1}\left[\begin{array}{ccc|c} 1/(s+1)^{n-r+1} & & & \\ & 1/(s+1) & & 0 \\ & & \ddots & \\ & & 1/(s+1) & \\ \hline & 0 & & 0_{p-r\times m-r} \end{array}\right]N^{-1},$$

$$U(s)=N\begin{bmatrix} 2^{n+k-r+1}s/(s+1)^k & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & \\ 0 & & \cdots & 0 \end{bmatrix}M.$$

Then

$$V(s)=(I-T(s)V(s))^{-1}T(s)=M^{-1}$$

$$\left[\begin{array}{ccc|c} (s+1)^k/((s+1)^{n+k-r+1}-2^{n+k}s) & & & \\ & 1/(s+1) & & 0 \\ & & \ddots & \\ & & 1/(s+1) & \\ \hline & 0 & & 0_{p-r\times m-r} \end{array}\right]N^{-1}$$

has characteristic polynomial $\Delta(s) = (s+1)^{r-1}((s+1)^{n+k-r+1} - 2^{n+k-r+1}s)$. Reasoning similarly as for part a), we conclude that A_{22} and F_{22} are strictly stable, (4) and (14) hold, (X, Y, Z) is controllable and observable, and X is nonsingular with a unit eigenvalue.

To complete the proof, we need only choose A_{12}, A_{21}, F_{12}, and F_{21} arbitrarily and solve for the remaining matrices from (3), (4) and (13), (14). □

Part 1) of Theorem 4 demonstrates that, in a certain sense, the high-frequency effects which bring about closed-loop instability do not correspond to the complement of a generic set, and hence cannot be dismissed as merely a pathological case.

V. Conclusions

We have shown that input-output information alone is insufficient for designing robust linear compensators. This conclusion leads one immediately to ask what further information is actually required to allow a robust design. Although we cannot give a clear answer yet, we can offer some insight. The development of our results indicates the high-frequency behavior in (2) and (12) plays a role in destabilization. Such behavior is closely related to the infinite-frequency structure of (2) and (12) with ϵ = 0 (see, e.g., [14]). One might therefore suspect that some knowledge of the poles and zeros at infinity in either the plant or compensator is essential. The exact form of such information and whether it can be easily measured are important topics for further research.

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A Nyquist-Type Stability Test for Multivariable Distributed Systems

D. W. LUSE

Abstract—**A multivariable Nyquist criterion is given which applies to a wide class of open-loop transfer matrices, including some which are improper or have infinitely many poles in the right-half plane. Application of the test to specific examples requires only elementary knowledge of complex variable theory.**

I. Introduction

The Nyquist test [6] for closed-loop stability of systems in terms of open-loop properties has been generalized for applicability to many types of feedback situations. For linear systems, the main three directions of research have been toward multivariable systems as originated by MacFarlane [1], distributed systems (e.g., [2]-[5]), and nonscalar gain variations (e.g., [7]). The more sophisticated stability tests often involve technical concepts such as the Fredholm index and almost periodic functions. The test given in this note applies to a very wide class of transfer matrices, but the stability conditions themselves involve only the winding numbers of simple, closed, piecewise-smooth curves. This allows a number of theoretical difficulties to be ignored in the theorem statement, and to be evaded in practice for many particular examples. These advantages are gained through two sacrifices of hypothesis. When stability of distributed systems is considered, the type (or types) of L_p stability is a concern [5], [10]. For this note, stability is restricted to be finite-gain L_2 stability. The second sacrifice is that the open-loop transfer matrix must be evaluated at some points in the open right-half plane.

Section II of this note states and briefly proves a version of the Nyquist criterion. Also included are four simple examples. Three of these are pathological examples to illustrate the strengths of the theory, while the fourth one shows why (idealized models of) feedback loops containing wave propagation behavior are not robust to small time delays.

II. Nyquist-Type Stability Test

A number of definitions from distribution theory are needed at this point. These allow for a brief and general theorem statement. The

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The author is with the Department of Electrical Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061.

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[7] J. D. Cobb, "A Geometric Decomposition Theorem for Singularly Perturbed Systems with Applications to Convergence of Solutions," *SIAM Conference on Control in the 90's*," San Francisco, May 1989.

ially stable semigroup. Special attention will be given to relations between the hypotheses used for the convergence results and convergence properties of the transfer functions for the approximating systems.

F. Kappel
Institute for Mathematics
University of Graz
Elisabethstrasse 16
A-8010 Graz
AUSTRIA

5:30 PM

Approximation techniques for parabolic control systems: a variational approach

In this paper we consider the linear quadratic regulator problem for a class of boundary control problems for parabolic systems. The problem is formulated using a variational approach and an approximation theory is developed for solutions of the associated operator Riccati equation. Our study includes strongly-damped elastic systems.

H.T. Banks
K. Ito
Center for Control Sciences
Brown University
Providence, RI 02912

6:00 PM

Uniform stabilization/exact controllability from the boundary of thermoelastic plates

We consider the small vibrations of a thin homogeneous, thermally and elastically isotropic plate of uniform thickness. It is known that thermal dissipation alone is sufficient to strongly, but not uniformly, stabilize the elastic components of the motion. It will be shown that theintroduction of additional dissipation through the action of bending and twisting moments, and shear force, at the boundary leads to uniform asymptotic stability and to explicit asymptotic energy estimates. The more difficult question of exact controllability of the elastic dynamics by means of such boundary forces and moments will also be considered. It may be shown that if the thermal diffusivity is sufficiently small, the reachable set of the elastic components of the motion is essentially the same as for purely elastic plate motion.

John E. Lagnese
Department of Mathematics
Georgetown University
Washington, DC 20057

MINISYMPOSIUM 10

Room: Cathedral Hill A

GEOMETRIC METHODS IN LINEAR CONTROL THEORY 2

Chair: Joyce O'Halloran

4:30 PM

On the Topological Structure of the Orbit Space of Controllable Generalized Linear Systems

Abstract: We study the topology of the orbit space of controllable descriptor systems modulo restricted system equivalence. Using a scaling action, we prove that this space is an analytic manifold. Using the Weierstrass decomposition, we obtain an analytic stratification of this manifold. By decomposing the strata into generalized Hermite cells, and using tools from Borel-Moore homology, we compute the singular homology groups for this space in the complex cases. Consequently, the orbit space of controllable descriptor systems is a smooth compactification of the orbit space of controllable state space systems modulo change of basis in the state space.

Uwe Helmke, Department of Mathematics, University of Regensburg, 8400 Regensburg, West Germany; Mark A. Shayman, Electrical Engineering Department and Systems Research Center, University of Maryland, College Park, Maryland 20742

5:00 PM

A Geometric Decomposition Theorem for Singularly Perturbed Systems with Applications to Convergence of Solutions

Let E and A be $n \times n$ matrix-valued analytic functions of a nonnegative real parameter ε with $sE(\varepsilon)-A(\varepsilon)$ a regular matrix pencil for all ε. The limiting behavior of the solutions of the family of differential equations

$$E(\varepsilon)\dot{x}=A(\varepsilon)x \qquad (1)$$

as $\varepsilon \to 0^+$ is at best only partially understood. Some sufficient conditions are known for certain types of convergence, but no necessary and sufficient conditions are known for nontrivial topologies. We first present a new geometric decomposition theorem for (1). We then show that this result enables us to prove a necessary and sufficient condition under which, for any initial vector, the corresponding solutions of (1) converge uniformly on compact subsets of $(0,\infty)$ with bounded peaking in $[0,1]$.

Daniel Cobb
Department of Electrical
and Computer Engineering
University of Wisconsin
Madison, WI 53706-1691